Project 2

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Contents

We are interested in studying the cohomology groups $H^*(\Gamma, V)$ for $\Gamma \subseteq \mathcal{G}(\mathbb{Q})$ arithmetic and V a finite dimensional algebraic representation of $\mathcal{G}$.

These cohomology groups have a geometric interpretation. If $K \subseteq \mathcal{G}(\mathbb{R})$ is a maximal compact subgroup, then $X = \mathcal{G}(\mathbb{R})/K$ is a real manifold and the symmetric space $X_\Gamma = \Gamma \setminus \mathcal{G}(\mathbb{R})/K$ is also a real manifold if Γ is a torsion free.

From $V(\mathbb{C})$, we can forms a local system $\tilde{V}$ on X_Γ. To do this, note that $\pi_1(X) = \Gamma$ and considering V as a representation of Γ, we can form the sheaf of sections $\tilde{V}$ of the bundle $G \times_\Gamma V \to \Gamma \setminus X$. This allows us to identify the cohomology of Γ with that of X_Γ:

$$H^*(\Gamma, V(\mathbb{C})) \cong H^*(X_\Gamma, \tilde{V}).$$

The group cohomology groups $H^*(\Gamma, V(\mathbb{C}))$ have a Hecke action which allows us to interpret them as the space of automorphic forms on G of level Γ. For example, in the case of modular forms, the Eichler-Shimura isomorphism demonstrates that we can reformulate the theory of weight k modular forms in terms of the cohomology of subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$:

(0.1) Theorem (Eichler-Shimura)**.** There is a Hecke-equivariant isomorphism

$$S_{k+2}(\Gamma) \oplus \overline{S_{k+2}}(\Gamma) \oplus \mathrm{Eis}_{k+2}(\Gamma) \xrightarrow{\sim} H^1(Y_\Gamma, \mathrm{Sym}^k(\mathbb{C}^2)).$$

This embedding gives us an idea of how one might generalise of the notion modular forms by replacing $\mathrm{SL}_2(\mathbb{Z})$ with other groups.

In the Eichler-Shimura isomorphism, the images of the cusp forms in the de Rham cohomology, compactly supported, and singular cohomology are all canonically isomorphic under Hecke-equivariant maps, so it is often convenient to consider instead the isomorphism $S_{k+2}(\Gamma) \oplus \overline{S_{k+2}(\Gamma)} \cong H^1_c(Y_\Gamma, \mathrm{Sym}^k(\mathbb{C}^2))$ with compactly supported cohomology. Similarly, if $\Gamma \setminus \mathcal{G}(\mathbb{R})$ is not compact modulo the center, then $H^*(X_\Gamma, \tilde{V})$ has subspaces corresponding to cuspidal automorphic forms and these are exhaustive, so we also need to consider other cohomology classes coming from cusp forms on the Levi's of proper parabolic subgroups, which correspond to the $\mathrm{Eis}_{k+2}(\Gamma)$ in the above.

We are interested in the case that X_Γ is compact. Then we have Matsushima's formula:

$$H^*(\Gamma, V) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma, V) H^*(\mathfrak{g}, K; H_\pi \otimes V)$$

where the sum is finite and runs over certain unitary representations (π, H_π) of $\mathcal{G}$. This allows us to reduce to the problem of studying the $H^*(\mathfrak{g}, K; H_\pi \otimes V)$, which we are able to calculate explicitly.

In this project, we will largely follow A.Borel and Wallach's book [1] 'Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups'. We will start by defining $(\mathfrak{g}, K)$-cohomology and all of the necessary terminology in the first section, and then prove some vanishing theorems for $(\mathfrak{g}, K)$-cohomology in the second section. In the third section, we will then prove Matsushima's formula. In the fourth, we shall prove a theorem which gives explicitly the Betti numbers of the $(\mathfrak{g}, K)$-cohomology of the tensor product of an irreducible finite dimensional complex representation of G with some induced representation of a parabolic, and show that this vanishes outside of an interval $[q_0, q_0 + l_0]$ for $l_0 = \mathrm{rk}(G) - \mathrm{rk}(K)$ and $2q_0 = \dim(G/K) - l_0$. In particular, we can deduce from this that for V a tempered $(\mathfrak{g}, K)$-module, $H^q(\mathfrak{g}, K; V \otimes F_\lambda) = 0$ for q outside of this interval. We conclude with a short discussion of Venkatesh's conjecure and explain why these results are indicative of it.

1 Preliminaries

1.1 Representations of Lie groups

(1.1) Notation. If $G, H, \ldots$ are real Lie groups then we will denote their Lie algebras by $\mathfrak{g}, \mathfrak{h}, \ldots$ If $\mathfrak{m}$ is a subspace of $\mathfrak{g}$, then we will denote its complexification by $\mathfrak{m}_c = \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$. A Lie algebra $\mathfrak{g}_0$ is called a ***real form*** of the complex Lie algebra $\mathfrak{g}$ if $\mathfrak{g} = (g_0)_c$. Using the correspondence between Lie groups and Lie algebras, we can then define the real form G_0 of a complex Lie group G.

Denote the universal enveloping algebra over $\mathbb{C}$ of $\mathfrak{g}$ by $U(\mathfrak{g})$ and its center by $Z(\mathfrak{g})$.

If G is a Lie group, let ${}^0G = \bigcap_{\chi \in X(G)} \ker |\chi|$ where $X(G)$ is the group of continuous homomorphisms $G \to \mathbb{R}^*$. It is normal and contains the derived group and all compact subgroups of G.

(1.2) Example. $\mathrm{SL}_2(\mathbb{R})$, the group of real 2×2 matrices of determinant 1, is a real compact simple Lie group with Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, the Lie algebra of real 2×2 traceless matrices. It is a real form for the complex Lie group $\mathrm{SL}_2(\mathbb{C})$.

Let (π, V) be a continuous G-module.

(1.3) Definition. Define a continuous map $c_v : G \to V$ by $c_v(g) = \pi(g)v$. Then for a continuous functional $\tilde{v}$ on V, we can define functions $c_{v,\tilde{v}}(g) = \langle c_v(g), \tilde{v} \rangle$, the ***coefficients*** of V.

If V is a Hilbert space, we can define the coefficients as $c_{v,w}(g) = (\pi(g)v, w)$ for $v, w \in V$ where $(\, , \,)$ is the scalar product on V.

$v \in V$ is ***differentiable*** if c_v is C^∞. Denote the space of such v by V^∞, which is a G-representation.

(1.4) Remark. $U(\mathfrak{g})$ acts on V^∞. To show this, suppose v is differentiable and define $f : \mathfrak{g} \to V$ by $f(X) = \pi(\exp(X))v$. This is differentiable. Define $\phi(X)v = f'(0)X$. Then $\phi(X) : V^\infty \to V^\infty$ and $\phi([X,Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X)$ so it gives an action of $U(\mathfrak{g})$ on V^∞.

(1.5) Definition. A vector $v \in V$ is ***G-finite*** if it is contained in a finite dimensional subspace stable under G. A G-module is ***locally finite*** if every element is G-finite.

Let K be a compact subgroup of G and W a finite dimensional K-module. Let $V_{(W)} = \mathrm{Im}(\mathrm{Hom}_K(W, V) \otimes W \xrightarrow{\tau \otimes w \mapsto \tau(w)} V)$. Then the space of K-finite vectors $V_{(K)}$ is given by $\bigcup_W V_{(W)}$ running over all such W.

If W is irreducible then we call $V_{(W)}$ the ***isotypic subspace*** of type W.

We say that V is ***admissible*** if all isotypic subspaces are finite dimensional.

Suppose K has been fixed. Then let $V_0 = V^\infty \cap V_{(K)}$, a space which is stable under $\mathfrak{g}$. Finite dimensional isotypic subspaces are contained in V^∞ and they are contained in V iff all isotypic subspaces are finite dimensional. In this case $V_0 = V_{(K)}$.

(1.6) Definition ((g, K)-modules)**.** A $(\mathfrak{g}, K)$***-module*** is a real or complex vector space which is a $\mathfrak{g}$-module and a locally finite and semisimple K-module such that the operations of $\mathfrak{g}$ and K satisfy the following compatibility conditions:

1. $k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$ for all $v \in V, k \in K, X \in \mathfrak{g}$.
2. If F is a K-stable finite dimensional subspace of V, then the representation of K on F is differentiable, and has differential $\pi|_{\mathfrak{k}}$. (That is, $\left(\frac{d}{dt}\exp(tY) \cdot v\right)|_{t=0} = Y \cdot v$ for all $v \in F$ and $Y \in \mathfrak{k}$.)

A $(\mathfrak{g}, K)$-module is ***admissible*** if it is admissible as a K-module.

(1.7) Definition. Suppose that V is a vector space satisfying (1) and (2) in (1.6) and in which every K-stable finite dimensional subspace is K-semisimple. Then $V_{(K)}$ is a $(\mathfrak{g}, K)$-module.

Given a $(\mathfrak{g}, K)$-module V, $\mathfrak{g}$ and K operate on its dual space V' and it satisfies the above, so we can define the ***contragradient*** $(\mathfrak{g}, K)$-module $\tilde{V} = (V')_{(K)}$ to V.

This gives us a functor from $\tilde{V}$ is admissible iff V is admissible.

(1.8) Definition. A $(\mathfrak{g}, K)$-module V is ***unitary*** if V is endowed with a positive non-degenerate scalar product $(\ ,\ )$ invariant under K and infinitesimally invariant under $\mathfrak{g}$:

1. $(k \cdot v, k \cdot w) = (v, w)$;
2. $(x \cdot v, w) + (v, x \cdot w) = 0$

for all $v, w \in V$, $k \in K$, $x \in \mathfrak{g}$.

(1.9) Remark. If G is a connected reductive Lie group with K maximal compact, and (π, V) is a continuous representation of G on a complex Hilbert space V. Then V is admissible iff $\pi|_K$ is unitary and each irreducible unitary representation of K occurs in it with finite multiplicity.

(1.10) Example. Irreducible unitary representations on Hilbert spaces are admissible ([4]8.1).

(1.11) Fact (The Peter-Weyl theorem)**.** [4]

1. The set of coefficients of G is dense in the space of continuous complex functions on G equipped with the uniform norm.
2. If V is a complex Hilbert space which is a unitary representation of a compact group G, then V admits an orthogonal direct sum decomposition into irreducible finite dimensional unitary representations of G.
3. $L^2(G)$ is a Hilbert space and is a unitary representation of G under the action $h \cdot f(g) = f(h^{-1}g)$. In its decomposition, the multiplicity of each irreducible representation is equal to its degree. That is, $L^2(G)$ is the closure of $\bigoplus_\pi H_\pi^{\oplus \dim H_\pi}$, summing over the set of isomorphisms classes of irreducible unitary representations of G.

(1.12) Remark. A corollary of the Peter Weyl theorem is that, if K is a compact topological group, then every irreducible unitary representation is finite-dimensional.

(1.13) Definition. A $(\mathfrak{g}, K)$-module (π, V) has ***infinitesimal character*** χ if there is a homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ such that $\pi(z) = \chi(z) \cdot \mathrm{Id}$ for all $z \in \mathcal{Z}(\mathfrak{g})$. This holds in particular if V is irreducible and admissible.

A $(\mathfrak{g}, K)$-module (π, V) has a ***central character*** ω_π if there is a character $\omega_\pi : Z(\mathfrak{g}, K) \to \mathbb{C}^*$ such that $\pi(z) = \omega_\pi(z) \cdot \mathrm{Id}$ for all $z \in Z(\mathfrak{g}, K)$, the subgroup of elements in the center of K acting trivially on $\mathfrak{g}$.

(1.14) Definition. Every irreducible admissible $(\mathfrak{g}, K)$-module can be realized as the space of K-finite vectors of an irreducible admissible differentiable G-module. Two smooth representations are ***infinitesimally equivalent*** if the two associated $(\mathfrak{g}, K)$-modules of K-finite vectors are isomorphic.

Let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G.

(1.15) Remark. Infinitesimally equivalent irreducible unitary representations are equivalent.

(1.16) Definition. If (π, V) is unitary and irreducible, then there exists a unitary character ω_π of $\mathcal{C}(G)$, the center of G. In this case, $|c_{u,v}|$ is a function on $G/\mathcal{C}(G)$ and we say V is in the ***discrete series*** if it is unitary, irreducible, and if its coefficients $c_{u,v}$ are square integrable on $G/\mathcal{C}(G)$.

If G is compact, then the final condition automatically holds.

We say that it is ***tempered*** if the coefficients are in $L^{2+\epsilon}(G/\mathcal{C}(G))$ for all $\epsilon > 0$.

Using [4] and some notes of Kevin Buzzard, we compute the irreducible admissible representations of $\mathfrak{gl}_2(\mathbb{R})$.

(1.17) Example (Classifying the irreducible $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2))$-modules)**.** Let $G = \mathrm{GL}_2$ over $\mathbb{R}$, so $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$, $K = O(2)$. Take $K_0 = \mathrm{SO}(2)$. Let V be an irreducible $(\mathrm{gl}_2(\mathbb{R}), \mathrm{SO}(2))$-module.

SO(2) is isomorphic to S^1 and every irreducible representation of this is a 1-dimensional representation $z \mapsto z^n$ for $n \in \mathbb{Z}$, so V is some direct sum of V_k where $V_k = \{v \in V \mid \gamma_\theta \cdot v = \gamma_{k\theta} v\}$ for $\gamma_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Now SO(2) acts on $\mathfrak{gl}_2(\mathbb{R})$ by conjugation. Take the standard basis for $\mathrm{gl}_2(\mathbb{C})$ comprising of $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We can replace it with $\alpha e \alpha^{-1}, \alpha f \alpha^{-1}, \alpha h \alpha^{-1}$ and relabel. Then $[h, f] = -2f$, $[e, f] = h$, $[h, e] = 2e$ and everything commutes with z as usual.

Let $\alpha = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ which has the property that $\alpha^{-1}\gamma_\theta\alpha = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ so acts nicely on the basis vectors, and $\gamma_\theta e \gamma_\theta^{-1} = e^{2i\theta} e$, $\gamma_\theta f \gamma_\theta^{-1} = e^{-2i\theta} f$, $\gamma_\theta h \gamma_\theta^{-1} = h$.

Further, noticing $h^2 = 1$ and $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$\begin{aligned} \exp(i\theta h) &= 1 + i\theta h + \frac{(i\theta h)^2}{2!} + \frac{(i\theta h)^2}{3!} + \cdots \\ &= \cos\theta + ih \sin\theta \\ &= \gamma_\theta \in K \end{aligned}$$

and as V is a $(\mathfrak{g}, K)$-module,

$$\begin{aligned} h \cdot v &= \left(\frac{d}{d\theta} \exp(\theta i h) \cdot v \right) |_{\theta=0} \\ &= \left(\frac{d}{d\theta} \gamma_{k\theta} v \right) |_{\theta=0} \\ &= ikv \end{aligned}$$

for all $v \in V_k$.

Now let H, E, F, X be the elements in the universal enveloping algebra corresponding to h, e, f, x. It can be checked on the basis vectors that (the Casimir element) $C = (H - 1)^2 + 4EF \in Z(\mathfrak{g})$, and so as V is irreducible, C acts by a scalar $\tilde{c}$. X also acts as a scalar $\tilde{x}$.

Notice $\gamma_\theta E v = (\gamma_\theta E \gamma_\theta^{-1})\gamma_\theta v$ so if $v \in V_k$, then $Ev \in V_{k+2}$, and similarly $Fv \in V_{k-2}$. Also, $EF = \frac{1}{4}(C - (H-1)^2)$, so $EFv = \frac{1}{4}(\tilde{c} - (k-1)^2)v$, and similarly $FEv = \frac{1}{4}(\tilde{c} - (k+1)^2)v$. So if $\tilde{c} \neq (k-1)^2$ and $v \neq 0$, then $Fv \neq 0$, and if $\tilde{c} \neq (k+1)^2$ and $v \neq 0$, then $Ev \neq 0$.

If there is $0 \neq v \in V_k$ with $Fv = 0$, then $\bigoplus_{r \geq 0} E^r \langle v \rangle$ is stable under E, F, H, X and K_0 so by irreducibility it is all of V. Also in this case, $\tilde{c} = (k-1)^2$. Similarly, if there exists

$0 \neq v \in V_k$ with $Ev = 0$, then $V = \bigoplus_{r \geq 0} F^r \langle v \rangle$ and $\tilde{c} = (k+1)^2$. Conversely, if $\tilde{c} = (k-1)^2$ and $V_k \neq 0$ then $FV_k = 0$, and if $\tilde{c} = (k+1)^2$ and $V_k \neq 0$ then $EV_k = 0$.

There are then only the following cases to consider:

1. $\tilde{c} \neq n^2$ and $V = \bigoplus_{k \in \mathbb{Z}} V_{2k}$;
2. $\tilde{c} \neq n^2$ and $V = \bigoplus_{k \in \mathbb{Z}} V_{2k+1}$;
3. $\tilde{c} = n^2$ for some $n \in \mathbb{Z}_{\geq 0}$ and $V = \bigoplus_{k \equiv n \mod 2} V_k$;
4. $\tilde{c} = n^2$ for some $n \in \mathbb{Z}_{\geq 0}$ and $V = \bigoplus_{k \not\equiv n \mod 2, k \leq -n-1} V_k$;
5. $\tilde{c} = n^2$ for some $n \in \mathbb{Z}_{\geq 0}$ and $V = \bigoplus_{k \not\equiv n \mod 2, 1-n \leq k \leq n-1} V_k$;
6. $\tilde{c} = n^2$ for some $n \in \mathbb{Z}_{\geq 0}$ and $V = \bigoplus_{k \not\equiv n \mod 2, n+1 \leq k} V_k$.

So a $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{SO}(2))$-module is determined by the infinitesimal characters $\tilde{c}, \tilde{x}$ and a ***type*** (an irreducible representation of K_0 which occurs as a subrepresentation of the $(\mathfrak{g}, K_0)$-module). In fact this correspondence is one to one.

This also proves that irreducibility implies admissibility in this case and that for fixed infinitesimal characters $\tilde{c}, \tilde{x}$, there are only finitely many $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{SO}(2))$-modules.

Now suppose that V is an irreducible $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2))$-module. V is either irreducible or reducible as a $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{SO}(2))$-module. Let $d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so $\mathrm{SO}(2)$ and $d\,\mathrm{SO}(2)$ are the two connected components of $\mathrm{O}(2)$.

If V is irreducible, then $dV_k = V_{-k}$ as $dHd^{-1} = -H$, so $V \neq \bigoplus_{k \equiv n \mod 2, k \leq -n-1} V_k$, $\bigoplus_{k \equiv n \mod 2, n+1 \leq k} V_k$.

For the remaining cases, either $V_0 \neq 0$ or $V_1 \neq 0$, and it is enough to find the d-action on these as the $V_k = \mathbb{C} v_k$ will be 1-dimensional.

If $\tilde{c} \in \mathbb{Z}^2$, then consider cases 3 and 5 and there are exactly two choices of d-action.

If $\tilde{c} \notin \mathbb{Z}^2$, then consider cases 1 and 2 and V is determined by $\tilde{x}$, $\tilde{c}$, a type, and one of the two choices of d-action.

If V is reducible, it will have a submodule W with $V = W \oplus dW$ and $dW \neq W$. This forces $\tilde{c} = n^2$ and $V = \bigoplus_{k \leq -n-1, n+1 \leq k, k \equiv n+1 \mod 2} V_k$.

1.2 Linear algebraic and reductive groups

(1.18) Definition. Let k be a field of characteristic 0 and K an algebraically closed extension of k. A subgroup $\mathcal{G}$ is ***linear algebraic*** if it is the vanishing of an ideal in $K[X_{11}, \ldots, X_{nn}]$.

It is ***defined over*** k if the ideal is generated by polynomials in $k[X_{11}, \ldots, X_{nn}]$. In this case we set $\mathcal{G}(k) = \mathcal{G} \cap \mathrm{GL}_n(k)$.

(1.19) Definition. If $K = \mathbb{C}$, $\mathcal{G}$ is a complex Lie group and if it is defined over $\mathbb{R}$, then $\mathcal{G}(\mathbb{R})$ is a Lie group.

We say that $\mathcal{G}$ is ***reductive*** if its Lie algebra is reductive (it is the direct sum of a semisimple and an abelian Lie algebra).

(1.20) Definition. A real Lie group G is ***reductive*** if there exists a linear algebraic group $\mathcal{G}$ defined over $\mathbb{R}$ whose identity component in the Zariski topology is reductive and there exists a morphism $\nu : G \to \mathcal{G}(\mathbb{R})$ with finite kernel and image an open subgroup of finite index in $\mathcal{G}(\mathbb{R})$.

(1.21) Definition. A subgroup of T is a ***torus*** if it is the inverse image of $\mathcal{S}(\mathbb{R})$ for $\mathcal{S}$ an $\mathbb{R}$-torus of $\mathcal{G}$. We say it is $\mathbb{R}$-***split*** if $\mathcal{S}$ is isomorphic to a product of finitely many copies of GL_1 with this isomorphism defined over $\mathbb{R}$.

The maximal $\mathbb{R}$-split tori of G are conjugate under G^0, the connected component of the identity in G, and their common dimension is $\mathrm{rk}_{\mathbb{R}}(G)$, the $\mathbb{R}$-***split rank of*** G.

The ***split component*** of G is the identity component of the greatest split torus in the center of G.

(1.22) Example. If $T = \mathbf{T}(\mathbb{R})$ for $\mathbf{T}$ a maximal torus over $\mathbb{R}$, then the connected component $T^0 = (\mathbb{R}_+)^a \times (S^1)^b$ for some $a, b \in \mathbb{N}$. The maximally split tori correspond to the single conjugacy class where a is maximal and the fundamental Cartan subgroups correspond to the class where b is maximal. Then A_G is the Lie algebra of the $(R_+)^a$ part of a fundamental Cartain subgroup.

(1.23) Definition. A ***Cartan involution*** θ of G is an involutive automorphism of G with fixed point set a maximal compact subgroup and which is the inversion on the split component of G. A ***Cartan involution*** on $\mathfrak{g}$ is an involutive automorphism such that for B the killing form, $B_\theta(X, Y) := -B(X, \theta Y)$ is positive definite.

Let G be a connected reductive Lie group and K a maximal compact subgroup and θ the Cartan involution associated to K. We have the ***Cartan decomposition***

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta(x) = -x\}$.

From now on, let B be a G and θ-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ with restriction to $\mathfrak{k}$ negative non-degenerate. If $\mathfrak{g}$ is semisimple, let this just be the killing form of $\mathfrak{g}$.

(1.24) Example. For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$, $\theta(X) = -X^T$ is a Cartan involution. Then $\mathfrak{p}$ is the subspace of symmetric matrices and $\mathfrak{k} = \mathfrak{so}_n(\mathbb{R})$ is the subspace skew-symmetric matrices.

(1.25) Remark. We have

$$B(\mathfrak{k}, \mathfrak{p}) = 0, \quad [\mathfrak{k}, \mathfrak{k}] = 0, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

and if $\mathfrak{k}$ contains no non-zero ideal of $\mathfrak{g}$, then $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.

(1.26) Definition. A closed subgroup $\mathcal{P}$ of $\mathcal{G}$ is ***parabolic*** if $\mathcal{G}/\mathcal{P}$ is a projective variety.

A ***parabolic subgroup*** P of G is the normalizer of a parabolic subgroup of $\mathfrak{p}$ of $\mathfrak{g}$ (a subalgebra containing a maximal solvable subalgebra). It is the inverse image of $\mathcal{P}(\mathbb{R})$ for some $\mathcal{P}$ a parabolic subgroup defined over $\mathbb{R}$ of $\mathcal{G}$.

The ***unipotent radical*** N of P is the analytic subgroup generated by the nilradical of $\mathfrak{p}$.

A ***Levi subgroup*** M of P is the inverse image of a Levi $\mathbb{R}$-subgroup $\mathcal{M}$ of $\mathcal{P}$. We have that $P = M \ltimes N$.

(1.27) Definition. A ***split component*** A of P is the split component of a maximal torus in the radical of P. In particular, A is a split component of $\mathcal{Z}_G(A)$ and $\mathcal{Z}_G(A)$ is a Levi subgroup of P and $P = MN = A \cdot {}^0M \cdot N$.

The ***parabolic rank*** $\mathrm{prk}(P)$ of P is defined to be $\dim A$.

A ***p-pair*** is a pair (P, A) consisting of a parabolic subgroup P and a split component A of P. If a minimal parabolic subgroup P_0 is fixed, the ***standard parabolic subgroups*** are defined to be the parabolic subgroups $P \supset P_0$. Similarly, if we have a minimal p-pair (P_0, A_0), a ***standard p-pair*** is (P, A) such that $P \supset P_0$ and $A \subset A_0$. It is ***semi-standard*** if only $A \subset A_0$.

Given P parabolic, let $\bar{P} = \theta(P)$ with $\bar{P} = M\bar{N}$ for $\bar{N} = \theta(N)$.

(1.28) Notation. Given a standard p-pair, let $\Phi(P, A)$ be the set of roots of P with respect to A. That is, the set of weights of A acting on $\mathfrak{n}$ via the adjoint action. Let $\Delta(P, A)$ be the set of simple roots (those not a sum of more than one element of $\Phi(P, A)$).

It is a fact that a choice of ordering on ${}_{\mathbb{R}}\Phi = \Phi(\mathfrak{g}_c, \mathfrak{a}_{0c})$ is equivalent to a choice of minimal parabolic subgroup $P_0 \supset A_0$ and ${}_{\mathbb{R}}\Phi^+ = \Phi(P_0, A_0)$. Having fixed an ordering, the fundamental highest weights ϖ_α for $\alpha \in \Delta$ are then defined by $(\varpi_\alpha, \beta) = \delta_{\alpha,\beta}(\beta, \beta)/2$ for $\alpha, \beta \in \Delta$ where $(\, , \,)$ is a scalar product invariant under the Weyl group W.

We will write $\Phi(P, A)$ for $\Phi(\mathfrak{p}, \mathfrak{a})$, not distinguishing between the global root and its differential at the origin.

1.3 Arithmetic groups

(1.29) Definition. If $\mathcal{G}$ is a reductive group defined over $\mathbb{Q}$ with subgroups Γ, Γ', then we say that Γ, Γ' are ***commensurable*** if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'.

$\Gamma \subset \mathcal{G}(\mathbb{Q})$ is ***arithmetic*** if after some (equivalently any) closed embedding $\mathcal{G} \hookrightarrow \mathrm{GL}_n$, Γ and $\mathrm{GL}_n(\mathbb{Z}) \cap \mathcal{G}(\mathbb{Q})$ are commensurable.

Such Γ are discrete in $\mathcal{G}(\mathbb{R})$, and are stable under $\mathcal{G}(\mathbb{Q})$-conjugation.

(1.30) Example.

- $\mathrm{SL}_n(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Q})$.
- $\Gamma_N = \ker(\mathrm{GL}_n(\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z}))$ is arithmetic, a ***congruence subgroup*** of level N.
- For $G = \mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2$, the usual $\Gamma_0(N), \Gamma_1(N)$ are arithmetic.

(1.31) Remark. If $\Gamma', \Gamma'' \subseteq \Gamma$ are finite index subgroups and there is an isomorphism $\varpi : \Gamma' \cong \Gamma''$, then we can define a Hecke operator $T_\varphi \in \mathrm{End}(H^q(\Gamma, V(\mathbb{C})))$ by

$$H^q(\Gamma, V(\mathbb{C})) \to H^q(\Gamma'', V(\mathbb{C})) \xrightarrow{\varphi^*} H^i(\Gamma', V(\mathbb{C})) \to H^q(\Gamma, V(\mathbb{C}))$$

where the first and second maps are restriction and corestriction.

(1.32) Example. In the case of $\mathcal{G} = \mathrm{SL}_2$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $\Gamma' = \Gamma' = \Gamma_0(p)$ and the isomorphism φ is conjugation by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and T_φ corresponds to the usual operator T_p.

(1.33) Definition. Let $\Gamma_1, \Gamma_2 \subset \Gamma$ be of finite index. Suppose T is a Hecke operator acting on $H^0(\Gamma, \mathbb{C})$. Restriction maps it to the same constant class in $H^0(\Gamma_2, \mathbb{C})$ and corestriction is multiplication by $[\Gamma : \Gamma_2] = [\Gamma : \Gamma_1]$, the ***degree*** of T. So T acts as multiplication by the degree.

Define

$$H^q(\Gamma, \mathbb{Q})_{\mathrm{inv}} = \{ s \in H^q(\Gamma, \mathbb{Q}) \mid Ts = \deg(T) s \, \forall T \}$$

so, in particular, $H^0(\Gamma, \mathbb{Q})_{\mathrm{inv}} = H^0(\Gamma, \mathbb{Q})$.

The tempered comohology is intended to be a kind of complement to this invarant cohomology. Let $H^q(\Gamma, \mathbb{Q})_{\text{temp}}$ be the largest $\mathbb{T}$-stable subspace W of $H^q(\Gamma, \mathbb{Q})$ such that for all $\epsilon > 0$ and $T \in \mathbb{T}$, $\lambda \leq c(\epsilon) \deg(T)^{1/2+\epsilon}$ for every eigenvalue λ of T on $W \otimes \mathbb{C}$.

(1.34) Remark. We show later that the tempered cohomology can be described quite explicitly. Its dimensions vanish are a sequence of binomial coefficients dependent on the associated symmetric space so vanish outside of an interval. Explictly, the tempered cohomology looks like the l_0-dimensional torus $(S^1)^{l_0}$ for $l_0 = \operatorname{rk} G - \operatorname{rk} K$, but shifted in such a way that the middle dimension is $(\dim G/K)/2$.

1.4 The Langlands classification

The Langlands classification over $\mathbb{R}$ states roughly that we can identify the irreducible admissible $(\mathfrak{g}, K)$-modules in terms of tempered representations of smaller groups.

(1.35) Definition. Let G be a real reductive group with maximal compact K. Fix (P_0, A_0) minimal and suppose (P, A) is a standard p-pair and $P = MN$ is the standard Levi decomposition decomposition.

Further, let σ be an irreducible tempered representation of ${}^0M$ and $\nu \in \mathfrak{a}_c^*$ be such that $\operatorname{Re}\langle \nu, \alpha \rangle > 0$ for $\alpha \in \Delta(P, A)$.

We refer to a triple (P, σ, ν) as ***Langlands data***.

We want a bijection between the set of Langlands triples and the set of irreducible admissible representations of G up to infinitesimal equivalence.

(1.36) Definition. Given an admissible finitely generated representation (σ, H_σ) of ${}^0M$, the induced representation $(\pi_{P,\sigma,\nu}, I_{P,\sigma,\nu})$ is the representation defined by right translations on

$$I_{P,\sigma,\nu} = \{f \in C^\infty(G; H_\sigma) \mid f(man \cdot g) = a^{(\rho_P+\nu)} \cdot \sigma(m) \cdot f(g) \, \forall g \in G, m \in {}^0M, a \in A, n \in N\}$$

where $a^\alpha := \alpha(a)$.

(1.37) Definition. If ν has $\operatorname{Re}\langle \nu, \alpha \rangle > 0$ for $\alpha \in \Phi(P, A)$, then define a a homomorphism of $(\mathfrak{g}, K)$-modules $j(\nu) : I_{P,\sigma,\nu} \to I_{\bar{P},\sigma,\nu}$ by $(j(\nu)f)(g) = \int_{\bar{N}} f(\bar{n}g) d\bar{n}$.

(1.38) Fact. If (σ, H_σ) is an irreducible tempered representation of ${}^0M$, then $j(\nu) I_{P,\sigma,\nu}$ is irreducible and in fact is equivalent to the unique irreducible quotient $J_{P,\sigma,\nu}$, the ***Langlands quotient***, of $I_{P,\sigma,\nu}$.

(1.39) Theorem (The Langlands classification)**.** Let (π, H) be an irreducible admissible representation of a real reductive group G. Then there exists a unique set of Langlands data (P, σ, ν) such that (π, H_0) is equivalent with $J_{P,\sigma,\nu}$ where H_0 is its space of K-finite vectors. .

(1.40) Fact. A representation (π, H) is tempered iff there is a standard p-pair (P, A), σ a discrete series representation of ${}^0M$, and $\nu \in i\mathfrak{a}^*$ such that (π, H_0) is equivalent to a $(\mathfrak{g}, K)$-module summand of $I_{P,\sigma,\nu}$.

(1.41) Example (Tempered representations of $\mathrm{SL}_2(\mathbb{R})$). Let $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$. Using the same notation as in (1.17), we can show that the Casimir element C acts by multiplication by some scalar $\tilde{c} = \mu^2$ and generates $Z(\mathfrak{g})$ so the infinitesimal character of any irreducible representation is determined entirely by $\mu \in \mathbb{C}$. The center of G is

$M = \{I, -I\}$ and acts either trivially or by multiplication by -1 on V. This means that the infinitesimal character and central characters can be identified with pairs (μ, ϵ) for $\mu \in \mathbb{C}$ and $\epsilon = \pm 1$. These allow us to group the discrete series representations into 'L-packets' corresponding to pairs (ϵ, μ).

Up to conjugacy, $\mathrm{SL}_2(\mathbb{R})$ has only one proper parabolic subgroup P_0, the upper triangular matrices with determinant 1. $P_0 = MAN$ where A is the set of diagonal matrices $\mathrm{diag}(a, b)$ with $a, b > 0$ and $ab = 1$, and N is the upper triangular matrices with 1's on the diagonal. Similarly, $\bar{P}$ are the lower triagular matrices and so on. So we need to consider only $P = G$ or $P = P_0$. The principal series representations are determined by $f : \mathbb{R}^\times \to \mathbb{C}^\times$ continuous which in turn are determined by μ and ϵ such that $f(tx) = t^\mu f(x)$, and $f(-x) = (-1)^\epsilon f(x)$ for $t > 0$, $x \in \mathbb{R}^\times$. Let $I_{\epsilon,\mu}$ be the corresponding representation after inducing to G. $I_{\epsilon,\mu}$ has central character ϵ and infinitesimal character μ.

Any irreducible admissible representation is an eigenspace of C and has an eigenvector for H so is a subrepresentation of one of the $I_{\epsilon,\mu}$.

Now as before, $I_{1,\mu}$ admits a basis $\{v_{2j}\}_{j\in\mathbb{Z}}$ and $I_{-1,\mu}$ a basis $\{v_{2j+1}\}_{j\in\mathbb{Z}}$, where $H(v_j) = jv_j$, $E(v_k) = \frac{\mu+j+1}{2} v_{j+2}$, $F(v_j) = \frac{\mu-j+1}{2} v_{j-2}$. Let $V_k = \langle v_k \rangle$.

Let $D_{+\mu} = \bigoplus_{j \geq \mu+1} V_j$ and $D_{-\mu} = \bigoplus_{j \leq -\mu-1} V_j$ for $\mu \in \mathbb{Z}$.

- $I_{\epsilon,\mu}$ is reducible iff $\mu \in \mathbb{Z}$ and $\epsilon = -(-1)^\mu$.
- $I_{-1,0} = D_{+0} \bigoplus D_{-0}$, a sum of of irreducible representations.
- If $I_{\epsilon,\mu}$ is reducible with $\mu > 0$ then it has a unique irreducible quotient which has dimension μ and kernel $D_\mu \oplus D_{-\mu}$.
- If $I_{\epsilon,\mu}$ is reducible with $\mu < 0$ then it has a unique irreducible subrepresentation which has dimension $-\mu$ and the quotient by it is $D_\mu \oplus D_{-\mu}$.
- $I_{\epsilon,\mu} \cong I_{\epsilon,-\mu}$ if it is irreducible.

So the irreducible admissible representations the finite dimensional representations of dimension μ for each $\mu \in \mathbb{Z}_{\geq 0}$ with central characters $\epsilon = -(-1)^\mu$, D_{+0} and D_{-0} with $\mu = 0$ and central characters $\epsilon = -1$, the D_μ for $\mu \in \mathbb{Z} \setminus \{0\}$ and central charaters $\epsilon = -(-1)^\mu$, and also $I_{\epsilon,\mu}$ for $\epsilon \neq -(-1)^\mu$.

By (1.40), the tempered representations are D_{+0}, D_{-0}, the D_k for $k \in \mathbb{Z} \setminus \{0\}$, and also $I_{1,i\mu}$ for $\mu \in \mathbb{R}$, and $I_{-1,i\mu}$ for $\mu \in \mathbb{R}^*$.

Now to apply the Langlands classification. For the already tempered representations we are done so can let $P = \mathrm{SL}_2(\mathbb{R})$. The finite dimensional representations and the $I_{\epsilon,\mu}$ for $\mathrm{Re}\,\mu > 0$, $\mu \notin \mathbb{Z}$ or $\epsilon \neq -(-1)^\mu$ are all irreducible quotients of $I_{\epsilon,\mu}$ for $\mathrm{Re}\,\mu > 0$ so are induced from tempered representations of P. In the remaining case, for $\mu \in \mathbb{Z}_{>0}$ and $\epsilon = -(-1)^\mu$, $I_{\epsilon,\mu}$ has a finite dimensional representation as its irreducible quotient so again corresponds to one of the tempered representations.

1.5 Relative Lie algebra cohomology

Let F be a commutative field, $\mathfrak{g}$ a finite dimensional Lie algebra over F, and $\mathfrak{k}$ a subalgebra of $\mathfrak{g}$. Let (π, V) be a $\mathfrak{g}$-module over F.

(1.42) Definition. We define

$$C^q = C^q(\mathfrak{g}; V) := \mathrm{Hom}_F(\bigwedge^q \mathfrak{g}, V) = \bigwedge^q \mathfrak{g}^* \otimes V$$

and $d : C^q \to C^{q+1}$ by

$$df(x_0, \ldots, x_q) = \sum_i (-1)^i x_i f(x_0, \ldots, \hat{x}_i, \ldots, x_q) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_q).$$

Then $d^2 = 0$ so we can define the ***Lie algebra cohomology groups*** $H^q(\mathfrak{g}; V)$ to be the cohomology of this complex.

(1.43) Definition. For $x \in \mathfrak{g}$, define $\theta_x : C^q \to C^q$ and $i_x : C^q \to C^{q-1}$ by

$$(\theta_x f)(x_1, \ldots, x_q) = \sum_i f(x_1, \ldots, [x_i, x], \ldots, x_q) + x f(x_1, \ldots, x_q)$$
$$(i_x f)(x_1, \ldots, x_{q-1}) = f(x, x_1, \ldots, x_{q-1}).$$

These satisfy the relation $\theta_x = d i_x + i_x d$.

Define
$$C^q(\mathfrak{g}, \mathfrak{k}; V) := \{ f \in C^q(\mathfrak{g}; V) \mid i_x f = \theta_x f = 0 \ \forall x \in \mathfrak{k} \} \leq C^q(\mathfrak{g}; V).$$

Then $C^q(\mathfrak{g}, \mathfrak{k}; V)$ is stable under d and we denote its cohomology groups $H^q(\mathfrak{g}, \mathfrak{k}; V)$, the ***relative cohomology groups*** of $\mathfrak{g}$ mod $\mathfrak{k}$ with coefficients in V.

(1.44) Remark. There is an identification

$$C^q(\mathfrak{g}, \mathfrak{k}; V) = \{ f \in \mathrm{Hom}_F(\bigwedge^q (\mathfrak{g}/\mathfrak{k}), V) \mid \sum_i f(x_1, \ldots, [x, x_i], \ldots, x_q) = x f(x_1, \ldots, x_q) \ \forall x \in \mathfrak{k}, x_i \in \mathfrak{g}/\mathfrak{k} \}.$$

In other words,
$$C^q(\mathfrak{g}, \mathfrak{k}; V) = \mathrm{Hom}_{\mathfrak{k}}(\bigwedge^q (\mathfrak{g}/\mathfrak{k}), V)$$
where the action of $\mathfrak{k}$ on $\bigwedge^q(\mathfrak{g}/\mathfrak{k})$ is induced by the adjoint representation.

(1.45) Fact (The Kunneth rule). Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, $V = V_1 \otimes V_2$, $\mathfrak{k}_i \subset \mathfrak{g}_i$, and the V_i are $\mathfrak{g}_i$-modules. Then $H^q(\mathfrak{g}, \mathfrak{k}; V) = \bigoplus_{a+b=q} H^a(\mathfrak{g}_1, \mathfrak{k}_1; V_1) \otimes H^b(\mathfrak{g}_2, \mathfrak{k}_2; V_2)$.

(1.46) Fact. Assume V, W are $\mathfrak{g}$ modules in perfect duality with respect to a $\mathfrak{g}$-invariant pairing $\langle \, , \, \rangle$ and $H^*(\mathfrak{g}, \mathfrak{k}; V)$, $H^*(\mathfrak{g}, \mathfrak{k}; W)$ are finite dimensional. Then $H^q(\mathfrak{g}, \mathfrak{k}; V) \cong (H^{m-q}(\mathfrak{g}, \mathfrak{k}; W))^*$ for all $q \in \mathbb{Z}$.

(1.47) Remark. Let $R = U(\mathfrak{g})$, $S = U(\mathfrak{k})$.

If V, W are $(\mathfrak{g}, K)$-modules, we can compute $\mathrm{Ext}^q_{(\mathfrak{g},K)}(V, W)$ by using a projective resolution

$$\cdots \to P_1 \to P_0 \to V$$

and then take $C^i = \mathrm{Hom}(P_i, W)$ and the cohomology of the resulting complex is $\mathrm{Ext}^q(V, W) = H^q(C^\bullet)$.

Let $P_i = R \otimes_S \bigwedge^i(\mathfrak{g}/\mathfrak{k})$ and define $\partial_q : P_q \to P_{q-1}$ by

$$\partial_q(r \otimes x_1 \wedge \cdots \wedge x_q) = \sum (-1)^{i-1} x_i \cdot r \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_q) \\ + \sum_{i<j} (-1)^{i+j} r \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_q).$$

Let $\epsilon : P_0 = R \otimes_S \mathbb{C} \to \mathbb{C}$ be the augmentation. This gives a projective resolution and this implies in particular that the functors $H^q(\mathfrak{g}, K; -)$ are the right derived functors of $W \mapsto \mathrm{Hom}_{\mathfrak{g},K}(\mathbb{C}, W)$.

1.6 $(\mathfrak{g}, K)$-cohomology

Now let $F = \mathbb{R}$, G be a Lie group with finite component group, and K be a maximal compact subgroup of G.

(1.48) Definition. For V be a $(\mathfrak{g}, K)$-module, define

$$C^q(\mathfrak{g}, K; V) := \mathrm{Hom}_K(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), V)$$

where K acts on $\mathfrak{g}/\mathfrak{k}$ via the adjoint representation.

Then

$$C^q(\mathfrak{g}, K; V) \subset C^q(\mathfrak{g}, K^0; V) = C^q(\mathfrak{g}, \mathfrak{k}, V)$$

so $C^q(\mathfrak{g}, K; V)$ is a subcomplex of $C^q(\mathfrak{g}, \mathfrak{k}; V)$. Define the $(\mathfrak{g}, K)$-***cohomology groups*** $H^q(\mathfrak{g}, K; V)$ to be the cohomology groups of this complex.

(1.49) Remark. K/K^0 acts on $C^q(\mathfrak{g}, \mathfrak{k}; V)$ and $C^q(\mathfrak{g}, K; V) = C^q(\mathfrak{g}, \mathfrak{k}; V)^{K/K^0}$, so

$$H^q(\mathfrak{g}, K; V) = H^q(\mathfrak{g}, \mathfrak{k}; V)^{K/K^0}.$$

(1.50) Example. Let $G = \mathrm{SL}_2(\mathbb{R})$ and $K = \mathrm{SO}_2(\mathbb{R})$. Take a basis $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Let V be an admissible representation. Now $C^q(\mathfrak{sl}_2, \mathfrak{so}_2, V) = \mathrm{Hom}_{\mathrm{so}_2}(\bigwedge^q(\mathfrak{sl}_2/\mathfrak{so}_2), V)$. Note that it is sufficient to check that functions are invariant under H.

$C^0(\mathfrak{sl}_2, \mathfrak{so}_2, V) = \mathrm{Hom}_h(\mathbb{R}, V) = \{v \in V \mid hv = 0\}$ and the same holds for $C^2(\mathfrak{sl}_2, \mathfrak{so}_2, V)$. If $g \in C^1(\mathfrak{sl}_2, \mathfrak{so}_2, V) = \mathrm{Hom}_h(\mathfrak{sl}_2/\mathfrak{so}_2, V)$, then if $g(e) = v$ and $g(f) = w$, $h \cdot v = g([h, e]) = g(2e) = 2v$ and $h \cdot w = -2w$ and this completely determines the action, so $C^1(\mathfrak{sl}_2, \mathfrak{so}_2, V) = \{v \in V \mid hv = 2\} \oplus \{w \in V \mid hw = -2\}$. So we want to find the cohomology of the complex

$$0 \to V^{h=0} \xrightarrow{v \mapsto (ev, fv)} V^{h=2} \oplus V^{h=-2} \xrightarrow{(v,w) \mapsto fv - ew} V^{h=0} \to 0.$$

For example, let V be the space of modular forms of weight k. We can calculate that $\exp(th) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ so for $f \in V$, $h \cdot f = \lim_{t \to 0}(\exp(th) \cdot f)|_{t=0}(z) = \lim_{t \to 0} \frac{d}{dt}(e^{-kt} f(e^{2t} z))|_{t=0}$. Now if $f \in V^{h=n}$ then this reduces to solving $(n + k) f(z) = 2z f'(z)$ and we get something of the form $cz^{\frac{n+k}{2}}$ (not completely sure on this line of argument). This is not a modular form if $|n| \geq k$ or if $n \not\equiv k \mod 2$, so if $k \geq 3$ or $k = 1$, then $V^{h=0} = V^{h=2} = V^{h=-2} = 0$ and $H^q(\mathfrak{sl}_2, \mathfrak{so}_2, V) = 0$ for all q. In particular, by the Kunneth rule, $H^q(\mathfrak{gl}_2, \mathfrak{k}, V) = 0$ also, so $H^q(\mathfrak{gl}_2, K, V) = (H^q(\mathfrak{gl}_2, \mathfrak{k}_2, V))^{K/K^0} = 0$.

1.7 The Laplacian

(1.51) Notation. Let $m = \dim \mathfrak{p}$, $n = \dim \mathfrak{g}$. Take an orthonormal basis $(x_i)_{1 \leq i \leq m}$ of $\mathfrak{p}$ and a pseudo-orthonormal basis $(x_a)_{m+1 \leq a \leq n}$ of $\mathfrak{k}$ with respect to B (i.e. $B(x_i, x_j) = \delta_{i,j}$ for $1 \leq i, j \leq m$ and $B(x_a, x_b) = -\delta_{ab}$ for $m + 1 \leq a, b \leq n$).

Make the convention that the indices i, j, k, l run from 1 to m, and a, b, c, d from $m + 1$ to n. By (1.25)

$$[x_i, x_j] = \sum_a c^a_{i,j} x_a, \quad [x_a, x_i] = \sum_j c^j_{a,i} x_j$$

and, using the invariance of B, $c^a_{i,j} = c^i_{aj}$.

(1.52) Definition. Let (y_s) be a basis of $\mathfrak{g}$ and let (y'_s) be its dual basis with respect to B. Then define the ***Casimir element*** $C = \sum_{1 \leq s \leq n} y_s \cdot y'_s \in Z(\mathfrak{g})$.

Notice that $C = \sum x_j^2 - \sum x_a^2$.

(1.53) Notation. Let A be a finite set. For $s \geq 0$, let $I_s = \{1, 2, \ldots, s\}$. Denote by (ω^a, ω^i) the basis of $\mathfrak{g}^*$ dual to (x_a, x_i) (ω^i can viewed as forming a dual basis to (x_i) of $\mathfrak{p}^*$).

If $I = \{j_1, \ldots, j_q\} \subset I_m$, let

$$\omega^I := \omega^{j_1} \wedge \cdots \wedge \omega^{j_q}$$

and for $\eta \in D^q(V) = \mathrm{Hom}_{\mathbb{R}}(\bigwedge^q \mathfrak{p}, V)$ let

$$d\eta(y_0, \ldots, y_q) = \sum_i (-1)^i y_i \cdot \eta(y_0, \ldots, \hat{y}_i, \ldots, y_q).$$

Define

$$\begin{aligned}\eta_I &= \eta_{x_{j_1}, \ldots, x_{j_q}} \\ &= \eta(x_{j_1}, \ldots, x_{j_q}) \\ &= (-1)^{u-1} \eta_{j_u, j_1, \ldots, \hat{j}_u, \ldots, j_q} \\ &= (-1)^{u-1} \eta_{j_u \cup I(u)}\end{aligned}$$

where $I(u) = I \setminus \{j_u\}$, so η can be written as

$$\begin{aligned}\eta &= \sum_{I \subset I_m, |I| = q} \eta_I \cdot \omega^I \\ &= \frac{1}{q!} \sum_{j_1, \ldots, j_q \in I_m} \eta_{j_1, \ldots, j_q} \omega^{j_1} \wedge \cdots \wedge \omega^{j_q}.\end{aligned}$$

Note also that

$$(d\eta)_I = \sum_{1 \leq u \leq q+1} (-1)^{u-1} \pi(x_u) \cdot \eta_{I(u)}.$$

(1.54) Remark. Let $(\tau, V) = (\rho \otimes \sigma, H \otimes E)$ for (ρ, E) a finite dimensional complex continuous representation of G and (σ, H) a unitary $(\mathfrak{g}, \mathfrak{k})$-module. There is an scalar product $(\ ,\ )_E$ on E which is invariant under $\mathfrak{k}$ and such that $\rho(x)$ is self adjoint for all $x \in \mathfrak{p}$ (an ***admissible scalar product***). This exists as E is a representation of the compact real form G_u so possesses a G_u-invariant Hermitian scalar product $(\ ,\ )_E$. $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ acts by skew-adjoint operators as if $x \in \mathfrak{g}_u$, then $(xv, v') + (v, xv') = 0$ and for $x = iy$ with $y \in \mathfrak{p}$, this gives $(yv, v') = (v, v')$ so $\mathfrak{p}$ is self-adjoint.

Then on $D^q(V) = \bigwedge^q \mathfrak{p}^* \otimes H \otimes E$ there is a scalar product which is the tensor product of the scalar product on $\bigwedge^q \mathfrak{p}^*$ given by B with $(\ ,\ )_V := (\ ,\ )_H + (\ ,\ )_E$. In particular, if $\mu = \sum_{I \subset I_m} \mu_I \omega^I$, $\eta = \sum_I \nu_I \omega^I$, then $(\mu, \nu) = \sum_I (\mu_I, \nu_I)_V$. These scalar products are invariant under $\mathfrak{k}$ and $(\theta_x \mu, \nu) + (\mu, \theta_x \nu) = 0$ for $x \in \mathfrak{k}$, $\mu, \nu \in D^q(V)$.

For $x \in \mathfrak{g}$, let $\tau(x)^*$ denote the adjoint of $\tau(x)$ with respect to $(\ ,\ )_V$. Then

$$\begin{aligned} \tau(x)^* &= -\tau(x) \quad \text{for } x \in \mathfrak{k}, \\ \tau(x)^* &= \rho(x) - \sigma(x) \quad \text{for } x \in \mathfrak{p}. \end{aligned} \tag{1}$$

(1.55) Definition. Let $\delta : D^q(V) \to D^{q-1}(V)$ be defined by

$$(\delta\eta)_J = \sum_{1 \le j \le m} \tau(x_k)^* \eta_{\{j\} \cup J}$$

for $J \subset I_m$ with $|J| = q - 1$.

Then δ commutes with θ_x for $x \in \mathfrak{k}$, maps $C^q(V)$ to $C^{q-1}(V)$, and is adjoint to d $((\delta\eta, \mu) = (\eta, d\mu))$.

(1.56) Definition. Define the ***Laplacian*** $\Delta = d\partial + \partial d$. It is an endomorphism of $D^q(V)$ for each q and it preserves $C^q(V)$.

We say that η is ***harmonic*** if $\Delta\eta = 0$ (equivalently, if $d\eta = \partial\eta = 0$, or equivalently if $(\Delta\eta, \eta) = 0$). Let $\mathcal{H}(V)$ be the space of harmonic forms on $C^q(V)$.

We say η is ***closed*** if $d\eta = 0$ and ***coclosed*** if $\partial\eta = 0$.

(1.57) Proposition. Suppose H is additionally an admissible $(\mathfrak{g}, \mathfrak{k})$-module.

For every q, the map $\mathcal{H}^q(V) \to H^q(V)$ is an isomorphism.

That is, every cohomology class is represented by a unique harmonic form.

Proof. This is equivalent to the orthogonal decomposition

$$C^q(V) = \mathcal{H}^q(V) \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(\partial)$$

which follows from elementary Hodge Theory and the fact that $C^q(V) = \mathrm{Hom}_{\mathfrak{k}}(\bigwedge^q \mathfrak{p} \otimes E^*, H)$ is finite dimensional. □

2 Some vanishing theorems

We will start by proving some results about the Laplacian and Casimir element. Suppose $\tau = \sigma \otimes \rho$ for σ a unitary $(\mathfrak{g}, \mathfrak{k})$-module and ρ a finite dimensional complex representation of G a connected Lie group.

We first prove a result in that if the Casimir element acts as scalar multiples of the identity under σ and ρ, if these scalars are equal then $H^q(\mathfrak{g}, \mathfrak{k}; \tau) = 0$ for all q, and otherwise all chains are closed and harmonic.

There is an equivalence between equivalence classes ω_Λ of irreducible square integral representations and regular roots. We use this in conjunction with Blattner's conjecture to prove that the lowest K-weight of ω_Λ is $\Lambda + \rho - 2\rho_k$ and that it has multiplicity one. For F irreducible, $V \in \omega_\Lambda$, and using the interpretation of $\mathrm{Ext}^i_{\mathfrak{g},\mathfrak{k}}(\mathfrak{g}, V_K)$ in terms of Yoneda extensions, we then prove that $\mathrm{Ext}^q_{\mathfrak{g},\mathfrak{k}}(F, V_K) = 0$ for all q if the highest weight of F is not $\Lambda - \rho$. If instead the highest weight is $\Lambda - \rho$ then C acts as $\langle \Lambda, \Lambda + 2\rho \rangle \cdot \mathrm{Id}$ on both F and H so, using the result about the Casimir element, $\mathrm{Ext}^q_{\mathfrak{g},\mathfrak{k}}(F, H) = H^q(\mathfrak{g}, \mathfrak{k}; F^* \otimes H) = C^i(\mathfrak{g}, \mathfrak{k}; F^* \otimes H)$. From the theorem of the highest weight and the result about the lowest weight, we deduce that $\dim \mathrm{Ext}^i_{\mathfrak{g},\mathfrak{k}}(F, V_K) = \delta_{i,q}$ for $2q = \dim G/K$ so vanishes for all but a specific valeu fo q.

If G is reductive and its identity component has compact center, then we can reduce to the case $G = G^0$ and decompose V and F into representations of the product of a semisimple group and a torus. We decompose $H^q(\mathfrak{g}, \mathfrak{k}, F)$ using the Kunneth rule and apply the results in the previous paragraph to prove that $\mathrm{Ext}^i_{\mathfrak{g},K}(F, V) = 0$ for $i \neq q$ with q as before. With the additional assumption that F is irreducible with respect to G^0, we go on to prove that $\dim H^q(\mathfrak{g}, K; V \otimes F) \leq 1$ at this q.

Finally, we prove Matsushima's vanishing theorem. This is the statement that, if H is a nontrivial irreducible admissible unitary $(\mathfrak{g}, \mathfrak{k})$-module, then $H^q(\mathfrak{g}, \mathfrak{k}; H) = 0$ for $\mathrm{q} \leq m(\mathfrak{g})$ where $m(\mathfrak{g}) \in \mathbb{N}$ is a constant dependent on $\mathfrak{g}$ which can be explicitly calculated.

2.1 Notation

Let G be a connected reductive Lie group with a maximal compact subgroup K. Let A_0 be a maximal connected commutative $\mathbb{R}$-split subgroup with Lie algebra orthogonal to that of $\mathfrak{k}$. Let P_0 be a minimal parabolic subgroup with split component A_0.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{a}_0$, and let H be the corresponding Cartan subgroup. If (P, A) is semi-standard, then

$$\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}$$

where $\mathfrak{b} = \mathfrak{b}_P = \mathfrak{h} \cap {}^0\mathfrak{m}$, a Cartan subalgebra of ${}^0\mathfrak{m}$. Also, $H = B \times A$ where $B = {}^0M \cap H$ is the corresponding Cartan subgroup of ${}^0M$.

$$\mathfrak{h}_c^* = \mathfrak{b}_c^* + \mathfrak{a}_c^*$$

where $\mathfrak{b}_c^*$ the space of linear forms on $\mathfrak{h}_0$ zero on $\mathfrak{a}$ and $\mathfrak{a}_c^*$ are those zero on $\mathfrak{b}$.

If (P, A) is semi-standard, then $\Phi(\mathfrak{m}_c, \mathfrak{h}_c) = \Phi({}^0\mathfrak{m}_c, \mathfrak{b}_c)$ may be identified with the set of roots zero on $\mathfrak{a}$ and if we let

$$2\rho = \sum_{\alpha \in \Phi^+} \alpha$$
$$2\rho_{0_M} = \sum_{\alpha \in \Phi({}^0\mathfrak{m}_c, \mathfrak{b}_c)^+} \alpha,$$

then $\rho|_{\mathfrak{b}} = \rho_{0_M}$.

(2.1) Definition. If $W = W(\mathfrak{g}_c, \mathfrak{h}_c)$ is the Weyl group of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$, and $W_M = W(\mathfrak{m}_c, \mathfrak{h}_c) = W({}^0\mathfrak{m}_c, \mathfrak{b}_c)$, then define $W^P = \{w \in W \,|\, w^{-1}(\alpha) > 0 \,\forall \alpha \in \Delta_M\}$, the set of right cosets of W_M in W.

Let $l(w)$ be the length of $w \in W$ with respect to the set of reflections $s_\alpha \in W$ for $\alpha \in \Delta$.

2.2 The Laplacian and Casimir element

(2.2) Proposition. Let $V = H \otimes E$ be as in (1.54). Then if $\eta \in C^q(V)$,

$$(\Delta_\tau \eta)_I = (\rho(C) - \sigma(C)) \cdot \eta_I$$

for $I \subset I_m$, $|I| = q$ and C the Casimir element.

Proof. Let $\pi = \sigma, \rho$, or τ. View V as a $(\mathfrak{g}, \mathfrak{k})$-module under π and let Δ_π be the corresponding Laplacian. Let $\eta \in D^q(V)$. Let u from from 1 to q and j from 1 to m. Let $I = \{j_1, \ldots, j_q\}$. Then

$$\begin{aligned}
(\partial d\eta)_I &= \sum_j \pi(x_j)^* (d\eta)_{j \cup I} \\
&= \sum_j \pi(x_j)^* \pi(x_j) \eta_I + \sum_{j,u} (-1)^u \pi(x_j)^* \pi(x_{j_u} \eta_{j \cup I(u)}) \\
(d\delta\eta)_I &= \sum_q (-1)^{u-1} \pi(x_{j_u}) (\partial\eta)_{I(u)} \\
&= \sum_{j,u} (-1)^{u-1} \pi(x_{j_u}) \pi(x_j)^* \eta_{j \cup I(u)}
\end{aligned}$$

and so

$$(\Delta_\pi \cdot \eta)_I = \sum_j \pi(x_j)\pi(x_j)^* \eta_I + \sum_{j,u} (-1)^{u-1} [\pi(x_{j_u}), \pi(x_j)^*] \eta_{j \cup I(u)}. \tag{2}$$

The first sum on the right hand side of (2) is equal to

$$\sum_j (\rho(x_j)^2 - \sigma(x_j)^2) \eta_I$$

so it remains to show that

$$\sum_a (\sigma(x_a)^2 - \rho(x_a)^2) \eta_I = \sum_{j,u} (-1)^{u-1} [\tau(x_{j_u}), \tau(x_j)^*] \eta_{j \cup I(u)}.$$

where $m + 1 \leq a \leq n$. Call the right hand side of this Q.

Let $\pi = \tau$. Then as $\sigma \otimes 1$ and $\rho \otimes 1$ commute,

$$\begin{aligned}
[\pi(x_{j_u}), \pi(x_j)^*] &= [\sigma(x_{j_u}) + \rho(x_{j_i}), \sigma(x_j)^* + \rho(x_j)^*] \\
&= [\sigma(x_{j_u}), \sigma(x_j)^*] + [\rho(x_{j_u}), \rho(x_j)^*].
\end{aligned} \tag{3}$$

Further, $\sigma(x_j)^* = -\sigma(x_j)$ and $\rho(x_j) = \rho(x_j)^*$, so

$$\pi(x_j)\pi(x_j)^* = \rho(x_j)^2 - \sigma(x_j)^2 = \rho(x_j)^*\rho(x_j) + \sigma(x_j)\sigma(x_j)^*.$$

It then follows from (2) that

$$\Delta_\tau = \Delta_\sigma + \Delta_\rho \tag{4}$$

on $D^q(V)$.

Using the formulas for $\tau(x)^*$ in (1) and (3),

$$\begin{aligned}[\tau(x_{j_u}, \tau(x_j)^*] &= [\sigma(x_j)\sigma(x_{j_u})] - [\rho(x_j), \rho(x_{j_u})] \\ &= \sum_{j,a} c^a_{j,j_u}(\sigma(x_a) - \rho(x_a)).\end{aligned}$$

Therefore

$$Q = \sum_a (\sigma(x_a) - \rho(x_a)) \underbrace{\Big(\sum_{j,u} (-1)^{u-1} c^a_{j,j_i} \eta_{j \cup I(u)}\Big)}_{L_a}.$$

But $c^a_{j,j_u} = c^j_{a,j_u}$, so

$$\begin{aligned}L_a &= \sum c^j_{a,j_u} \eta(x_{j_1}, \ldots, x_j, \ldots, x_{j_q}) \\ &= \sum_u \eta(x_{j_1}, \ldots, [x_a, x_{j_u}], \ldots, x_{j_q}) \\ &= -(\theta_{x_a}\eta)(x_{j_1}, \ldots, x_{j_q}) + \tau(x_a)\eta(x_{j_1}, \ldots, x_{j_q}) \\ &= \tau(x_a)\eta_I\end{aligned}$$

as $\eta \in C^q(V)$ is annihilated by θ_x for $x \in \mathfrak{k}$. So we are done. □

(2.3) Corollary. Let $\eta \in D^q(V)$. Then $\Delta_\tau \eta = 0$ iff $\Delta_\rho \eta = \Delta_\sigma \eta = 0$.

Proof. This follows from (4) and as $\Delta\eta = 0$ iff $(\Delta\eta, \eta) = 0$. □

(2.4) Corollary. Assume $\sigma(C) = s \cdot \mathrm{Id}$, $\rho(C) = r \cdot \mathrm{Id}$.

(a) If $r \neq s$, then $H^q(\mathfrak{g}, \mathfrak{k}; H \otimes E) = 0$ for all q

(b) If $r = s$, then all cochains are closed, harmonic, and we have

$$H^q(\mathfrak{g}, \mathfrak{k}; H \otimes E) = C^q(\mathfrak{g}, \mathfrak{k}; H \otimes E) = \mathrm{Hom}_{\mathfrak{k}}(\bigwedge^q \mathfrak{p}; H \otimes E)$$

for all q.

Proof.

(a) $\Delta = (r-s)\,\mathrm{Id}$ on $C^q(H \otimes E)$ for all q. If η is a q-cocycle then $d\eta = 0$, so $\Delta\eta = d\partial\eta$, and so $\eta = (r-s)^{-1}\Delta\eta = (r-s)^{-1}d\partial\eta$ which is a coboundary.

(b) In this case $\Delta = 0$ so all cochains are harmonic so closed and coclosed as $\Delta\eta = 0$ iff $d\eta = \partial\eta = 0$.

□

2.3 Cohomology with respect to square integrable coefficients

(2.5) Remark. Harish-Chandra classified the discrete series representations of connected semisimple groups and showed that such a group G has a discrete series iff T is a Cartan subgroup of G.

Let G be semisimple and let T be a maximal torus in K. Suppose that T is a Cartan subgroup of G.

Let F be a finite dimensional $\mathfrak{g}$-module and $H = V_K$ for V a square integrable representation of G. If $H \neq 0$, then V is also a representation for G_0 with Lie algebra $\mathfrak{g}_c$.

(2.6) Remark (Highest Weight Theorem)**.** Let Φ (resp. Φ_k) be a root system of $\mathfrak{g}_c$ (resp.$\mathfrak{k}_c$) with respect to $\mathfrak{t}_c$. Let $W = N_G(T)/T$ (resp. W_k), the Weil group.

Let $P(\Phi) \subset \mathfrak{t}^*$ (resp. $P(\Phi_k)$) be the lattice of weights of Φ (resp. Φ_k) (i.e. the set of $\lambda \in \mathfrak{t}^*$ such that $2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle \in \mathbb{Z}$ for all simple roots α where $\langle\,,\,\rangle$ is a W-invariant scalar product (say, given by the killing form) on $\mathfrak{t}^*$). $P(\Phi)$ forms a lattice in $i \cdot \mathfrak{t}^*$ (the space of functions in $\mathfrak{t}_c^*$ taking imaginary values on $\mathfrak{t}$) and the killing form of $\mathfrak{g}$ induces an inner product on it.

Let $\Lambda \in P(\Phi)$ and call it ***regular***. Define

$$\begin{aligned} P^+ &= P(\Phi)^+ = \{\alpha \in P(\Phi) \mid \langle\Lambda,\alpha\rangle > 0\} \\ P_k^+ &= P(\Phi_k) \cap P^+ \\ \Phi^+ &= \Phi \cap P^+ \\ \Phi_k^+ &= \Phi_k \cap P^+. \end{aligned}$$

We can endow $i\mathfrak{t}^*$ with an ordering by setting $\lambda > \mu$ if $\lambda - \mu$ is a linear combination of roots in Φ^+ with non-negative coefficients. So we can define the highest weight of a representation V of $\mathfrak{g}$.

The theorem of the highest weight states that if $\mathfrak{g}$ is finite dimensional and complex, then every finite dimensional irreducible representation has a unique highest weight which is dominant integral (so in the same orbit of W as Λ), and any two such representations with the same highest weight are isomorphic. Further, for each dominant integral λ there is a finite dimensional irreducible representation with highest weight λ. In other words, there is a correspondence:

$$\left\{\begin{matrix}\text{equivalence classes of irreducible} \\ \text{square integrable representations of } G\end{matrix}\right\} \leftrightarrow \{\text{orbits of } W_k\} \subset \{\text{regular elements in } P(\Phi)\}$$

$$\omega_\Lambda \leftrightarrow \Lambda.$$

The elements of the class of representation ω_Λ have infinitesimal characters χ_Λ. In particular, χ_Λ is the infinitesimal character of the finite dimensional irreducible representation with highest weight $\Lambda - \rho$.

(2.7) Notation. If $\mu \in P(\Phi_k)$, let F_μ denote an irreducible representation with highest weight μ.

We will assume the following [3]:

(2.8) Theorem (Blattner's conjecture)**.** Suppose G has a faithful finite dimensional representation. If μ is dominant with respect to Φ_k^+, then the irreducible K-module of highest weight μ occurs in the discrete series representation ω_Λ with multiplicity

$$\sum_{w \in W_K} \operatorname{sgn}(w) S(w(\mu + \rho_k) - \Lambda - \rho + \rho_k)$$

where $S(\mu)$ is the number of distinct ways that μ can be written as a sum of positive roots not in Φ_k.

(2.8) implies:

(2.9) Proposition. Let $\Lambda \in P(\Phi)$ be regular and $(\pi, V) \in \omega_\Lambda$. Then

1. $\dim \operatorname{Hom}_K(F_{\Lambda+\rho-2\rho_k}, V) = 1$,
2. if $\operatorname{Hom}(F_\mu, V) \neq 0$ with $\mu \in P_K^+$, then $\mu = \Lambda + \rho - 2\rho_k + Q$ where Q is a sum of elements in Φ^+.

That is, the lowest K-weight of ω_Λ is $\Lambda + \rho - 2\rho_k$ and it is of multiplicity one (using Schur's lemma).

(2.10) Theorem. Let (σ, F) be an irreducible finite dimensional represention of G, and let $(\pi, V) \in \omega_\Lambda$.

(a) If the highest weight of (σ, F) relative to Φ^+ is not $\Lambda - \rho$, then $\operatorname{Ext}^i_{\mathfrak{g},\mathfrak{k}}(F, V_K) = 0$ for all i

(b) If the highest weight is $\Lambda - \rho$, then $\dim \operatorname{Ext}^i_{\mathfrak{g},\mathfrak{k}}(F, V_K) = \delta_{i,q}$, where $q = (\dim G/K)/2$.

Proof.

(a) We will first show that if U is finite dimensional and $\chi_{\tilde{U}} \neq \chi_V$, then $H^q(\mathfrak{g}, \mathfrak{k}; U \otimes V) = 0$ for all q.

$$H^q(\mathfrak{g}, \mathfrak{k}, U \otimes_F V) = H^q(\mathfrak{g}, \mathfrak{k}, \operatorname{Hom}_F(U', V)) = \operatorname{Ext}^q(F, \operatorname{Hom}_F(U', V)) = \operatorname{Ext}^q(U', V)$$

where F is the field $\mathfrak{g}$ is over ($\mathbb{R}$ or $\mathbb{C}$).

Note that if U', V are $(\mathfrak{g}, \mathfrak{k})$-modules, we can interpret $\operatorname{Ext}^q(U', V)$ as equivalence classes (up to morphisms of chain complexes which are the identities on both ends) of exact sequences

$$S : 0 \to V \to E_{q-1} \to \cdots \to E_0 \to U' \to 0$$

in the category $\mathcal{C}$ of $(\mathfrak{g}, \mathfrak{k})$-modules, a commutative group under the Baer sum.

Let $S_q(U', V)$ be the set of such sequences. If $\chi'_U \neq \chi_V$, there is $z \in Z(\mathfrak{g})$ with $\chi_V(z) = 1$, $\chi_{U'}(z) = 0$. Take some $S \in S_q(U', V)$. Then z acts on each term of the sequence S and defines an endomorphism $\gamma(z)$ of S. Then clearly $\gamma(z)_V = 1$ and $\gamma(z)_{U'} = 0$

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & V & \longrightarrow & E_{q-1} & \longrightarrow & \cdots & \longrightarrow & E_0 & \longrightarrow & U' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \gamma(z)_{q-1}} & & & & \downarrow{\scriptstyle \gamma(z)_0} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & V & \longrightarrow & E_{q-1} & \longrightarrow & \cdots & \longrightarrow & E_0 & \longrightarrow & U' & \longrightarrow & 0 \end{array}$$

so $S \equiv 0$ and we must have $\mathrm{Ext}_q(U', V) = 0$.

$\chi_{U'} = \chi_{\tilde{U}} \neq \chi_V$ so the result now follows.

Finally, if the highest weight of F is not $\Lambda - \rho$, then $\chi_{\tilde{V_K}} \neq \chi_F$ so $\mathrm{Ext}^q(F, V_K) = 0$.

(b) We will first show

$$\mathrm{Ext}^i_{\mathfrak{g},\mathfrak{k}}(F, H) = H^i(\mathfrak{g}, \mathfrak{k}, F^* \otimes H) = \mathrm{Hom}_{\mathfrak{k}}(\bigwedge^i \mathfrak{p} \otimes F, H) \tag{5}$$

where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition.

To prove this, note that the Casimir element C is central so acts on simple modules by a scalar and acts on the finite dimensional highest weight module of weight Λ by the constant $\langle \Lambda, \Lambda + 2\rho \rangle$. Then $\sigma(C) = \pi(C) = r \cdot \mathrm{id}$ as they have the same highest weight. Then by (2.4) all cochains are closed and harmonic so we are done.

Let $\Phi_n = \Phi - \Phi_k$ and $\Phi_n^+ = \Phi_n \cap \Phi^+$, and let $\rho_n = \rho - \rho_k$. The weights of T on $\bigwedge^i \mathfrak{p}_c$ are of the form $\alpha_1 + \cdots + \alpha_i$ with α_j distinct in Φ_n so are of the form $\alpha_1 + \cdots \alpha_j - \alpha_{j+1} - \cdots - \alpha_i$ for $\alpha_1, \ldots, \alpha_j, -\alpha_{j+1}, \ldots, -\alpha_i$ distinct elements of Φ_n^+.

$\alpha_1 + \cdots + \alpha_j = 2\rho_n - \gamma_1 - \cdots - \gamma_t$ for $\gamma_k \in \Phi_n^+$ distinct as $2\rho_n = \sum_{\alpha \in \Phi_n^+} \alpha$. So the weights of $\bigwedge^i \mathfrak{p}_c$ are of the form $2\rho_n - Q$ for Q a sum of elements in Φ_n^+.

If $2\rho_n$ is a weight in $\bigwedge^i \mathfrak{p}_c$, then i must equal the number of terms in $2\rho_n$ so $i = |\Phi_n^+| = \frac{1}{2}|\Phi_n| = (\dim G/K)/2 = q$. So, by the theorem of the highest weight, $2\rho_n$ has multiplicity 1

The weights of (σ, F) relative to T are of the form $\Lambda - \rho - Q$ with Q a sum of weights in Φ^+ as it has highest weight $\Lambda - \rho$, so $\Lambda - \rho$ also has multiplicity 1 by the theorem of the highest weight.

This implies that the weights of T on $\bigwedge^i \mathfrak{p}_c \otimes F$ (which are the sum of the weights of F and $\bigwedge^i \mathfrak{p}_c$) are of the form $2\rho_n + \Lambda - \rho - Q$. If $2\rho_n + \Lambda - \rho + Q'$ is a weight of $\bigwedge^i \mathfrak{p}_c \otimes F$, then $Q' = 0$, $i = q$, and the weight $\Lambda + \rho - 2\rho_n$ has multiplicity 1 in $\bigwedge^i \mathfrak{p}_c \otimes F$.

This then implies that if λ is Φ_k^+-dominant integral and $\mathrm{Hom}_K(F_\lambda, \bigwedge^i \mathfrak{p}_c \otimes F) \neq 0$, then $\lambda = \Lambda + \rho - 2\rho_k - Q$. If $\lambda = \Lambda + \rho - 2\rho_k + Q$, then $\mathrm{Hom}_K(F_\lambda, \bigwedge^i \mathfrak{p}_c \otimes F) \neq 0$ implies $Q = 0$ and $i = q$ and $\dim \mathrm{Hom}_K(F_{\Lambda+\rho-2\rho_k}, \bigwedge^q \mathfrak{p}_c \otimes F) = 1$.

By (2.9) and above

$$\begin{aligned}
\dim \operatorname{Ext}^i_{\mathfrak{g},\mathfrak{k}}(F,H) &= \dim \operatorname{Hom}_{\mathfrak{k}}(\bigwedge^i \mathfrak{p}\otimes F, H)\\
&= \sum_{\lambda\in\Phi} \dim \operatorname{Hom}_K(\bigwedge^i \mathfrak{p}_c \otimes F, F_\lambda)\\
&= \sum_{Q \text{ sum in } \Phi^+} (\dim \operatorname{Hom}_K(\bigwedge^i \mathfrak{p}_c\otimes F, F_{\Lambda+\rho-2\rho_k+Q}) + \dim \operatorname{Hom}_K(\bigwedge^i \mathfrak{p}_c\otimes F, F_{\Lambda+\rho-2\rho_k-Q}))\\
&= \delta_{i,q}
\end{aligned}$$

as $\dim \operatorname{Hom}_K(\bigwedge^i \mathfrak{p}_c \otimes F, F_{\Lambda+\rho-2\rho_k+Q} = 0$ for $Q\neq 0$ with either Q or $-Q$ a sum in Φ^+ by (2.9) and above.

□

(2.11) Remark. We can show similarly to in part (a) above that if U, V are $(\mathfrak{g}, K)$-modules with infinitesimal character χ_U, χ_V, (resp. central characters ω_U, ω_V), then

1. If $\chi_U \neq \chi_V$ (resp. $\omega_U \neq \omega_V$), then $\operatorname{Ext}^q_{\mathfrak{g},K}(U,V) = 0$ for all q.
2. If U is finite dimensional and $\chi_{\tilde{U}} \neq \chi_V$ (resp. $\omega_{\tilde{U}} \neq \omega_V$), then $H^q(\mathfrak{g}, K; U\otimes V) = 0$ for all q.

(2.12) Lemma. Let L be a reductive group, L' a normal open subgroup of L, and (π, V) an irreducible admissible L-module. Then V is a direct sum of finitely many irreducible L'-modules.

Proof. Let $Q \subset L$ be a maximal compact subgroup and $Q' = Q \cap L'$. Then Q' is a maximal compact subgroup of L' and Q' is normal and finite index in Q.

Let $\sigma \in \hat{L}' = \{$equivalence classes of irreducible unitary representations of $L'\}$. By Frobenius reciprocity, if τ restricts to σ then $\operatorname{Hom}_L(\operatorname{Ind}^L_{L'}\sigma, \tau) \cong \operatorname{Hom}_{L'}(\sigma,\sigma)$ which is 1-dimensional, so τ must be isomorphic to one of the irreducible submodules of $\operatorname{Ind}^L_{L'}(\sigma)$ and so there are finitely many choices of $\tau \in \hat{L}$ restricting to σ. Therefore V is an admissible L'-module.

It suffices to show that the existence of one irreducible L'-submodule $U \subset V$ as then $V = \sum_{x\in L/L'} xU$ and hence a direct sum of finitely many such transforms.

Let $(\tilde{\pi}, \tilde{V})$ by the contragradient representation. It is a simple L-module and hence is a finitely generated L'-module. Consequently it has a proper simple quotient. But V is an irreducible L-module so $\tilde{V}$ is infinitesimally equivalent to $\tilde{V}$ and so V has a proper simple L'-module U. □

(2.13) Lemma. Let L, L' be as in the previous lemma and K a maximal compact subgroup of L and $K' = K\cap L'$ Let (π, E) be an irreducible admissible $(\mathfrak{l}, K)$-module. Then E is the direct sum of finitely many irreducible admissible $(\mathfrak{l}, K')$-modules.

Proof. It is a fact that E may be viewed as the module of K-finite vectors of an irreducible admissible smooth L-module E_1. Then by the previous lemma, E_1 is a direct sum of finitely many irreducible admissible L'-modules.

The module of K-finite vectors of an irreducible smooth L'-module is irreducible, so we can take the K-finite vectors of each of the summands of E_1 to get E as a direct sum of finitely many irreducible $(\mathfrak{l}, K')$-modules. $\square$

(2.14) Theorem. Let M be a reductive group whose identity component has a compact center. Let (π, V) be a discrete series representation of M and (σ, F) a finite dimensional irreducible representation of M. Let $q = (\dim M/K)/2$.

Then $\mathrm{Ext}^i_{\mathfrak{g},K}(F, V) = 0$ for $i \neq q$.

Proof. The restriction of (π, V) to M^0 is the direct sum of finitely many irreducible representations by above, and these are square integrable (as V has square integrable coefficients and so the restriction does also). By (1.49) we can reduce to the case that M is connected and equal to its identity component.

Then as M is connected and reductive, $M = M' \cdot S$ where M' is semisimple and S is a central torus.

We may view V and F as $M' \times S$ modules. Then V is the tensor product of a one-dimensional representation of S (as S abelian) by an irreducible representation of M', which is also square integrable.

Since F is finite dimensional, it is a direct sum of irreducible representations, each of which has the form $F_S \otimes F_{M'}$ for F_S a one-dimensional S-representation and $F_{M'}$ an irreducible M'-representation. We can assume that F is an irreducible finite dimensional representation of M' semisimple. Using the Kunneth rule

$$H^q(\mathfrak{m}, \mathfrak{k}; F) = \bigoplus_{a+b=q} H^a(\mathfrak{m}', \mathfrak{k}'; F_{M'}) \otimes H^b(\mathfrak{s}, \mathfrak{k}; F_S).$$

Notice that $q = (\dim M/K)/2 = (\dim M'S/K)/2 = (\dim M'/(M' \cap K))/2$ and the rest follows from (2.10).

$\square$

(2.15) Proposition. With the assumptions of the (2.14), and also assuming that F is irreducible with respect to M^0,

$$\dim H^q(\mathfrak{m}, K; V \otimes F) \leq 1$$

for $q = q(G)$.

Proof. Let V_0 be an irreducible $(\mathfrak{m}, K^0)$-submodule of V. Let U be the $(\mathfrak{m}, K)$-module induced from $(\mathfrak{m}, K^0)$-module V_0. As a vector space, $U = I^K_{K^0}(V_0)$, and $U \otimes F$ may be thought of as a $(\mathfrak{m}, K)$-module induced from $V_0 \otimes F$. By Frobenius reciprocity, we have an exact sequence of $(\mathfrak{m}, K)$-modules

$$0 \to V \otimes F \to U \otimes F \to V' \otimes F \to 0 \tag{6}$$

and moreover,

$$\mathrm{Hom}_K(\bigwedge^i(\mathfrak{m}/\mathfrak{k}), U \otimes F) = \mathrm{Hom}_{K^0}(\bigwedge^i(\mathfrak{m}/\mathfrak{k}), V_0 \otimes F)$$

so

$$H^i(\mathfrak{m}, K; U \otimes F) = H^i(\mathfrak{m}, K^0; V_0 \otimes F). \tag{7}$$

Let $W = V, V'$. Then

$$H^i(\mathfrak{m}, K; W \otimes F) = (H^i(\mathfrak{m}, K^0; W \otimes F))^{K/K^0}.$$

Since W is a direct sum of finitely many discrete series representations of M^0, (2.14) shows that

$$H^i(\mathfrak{m}, K; W \otimes F) = 0$$

for $i \neq q(G)$.

Hence, taking the cohomology in (6) gives an embedding

$$0 \to H^q(\mathfrak{m}, K; V \otimes F) \to H^q(\mathfrak{m}, K; U \otimes F)$$

for $q = q(G)$.

So now we are reduced to the case $U = V$. (7) allows us to further assume that M is connected so we can write $M = M' \cdot S$ for S compact commutative and M' simple connected. Then V_0 and F are the tensor products of a 1-dimensional representations of S by irreducible representations of M' so we can apply (2.10) and the Kunneth rule. □

2.4 Matsushima's vanishing theorem

If $L(\,,\,)$ is the symmetric bilinear form on $\mathfrak{k}$ defined by

$$L(x, y) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{p}}\, x \circ \mathrm{ad}_{\mathfrak{p}}\, y),$$

then

$$B(x, y) = B_{\mathfrak{k}}(x, y) + L(x, y).$$

The eigenvalues of $\mathrm{ad}\, x$ for $x \in \mathfrak{k}$ are purely imaginary and $\mathfrak{k}$ acts faithfully on $\mathfrak{p}$ so $L(\,,\,)$ is negative and non-degenerate.

Let

$$A = \min_{x \in \mathfrak{k}, B(x) = -1} -L(x, x) \in (0, 1].$$

Then using $c^a_{i,j} = c^i_{a,j}$,

$$L(x_a, x_b) = \sum_{i,j} c^i_{aj} \cdot c^k_{bi} = \sum_{i,j} c^a_{ij} \cdot c^b_{ji}$$
$$L(x_a, x_b) = -\sum_{ij} c^a_{ij} \cdot c^b_{ij}.$$

Assume that the x_a's form an orthogonal basis with respect to $L(\,,\,)$. Set

$$R(x, y) = -\,\mathrm{ad}[x, y]_{\mathfrak{p}}$$

so $R(x,y)\cdot z = [[y,x],z]$ for $x,y,z \in \mathfrak{p}$ and define

$$R_{ijkl} = B([[x_l,x_k],x_j],x_i) = B([x_l,x_k],[x_j,x_i])$$

so

$$R_{ijkl} = -\sum_a c^a_{kl}\cdot c^a_{ij}.$$

We denote by η_{ij} the coordinates of an element $\eta \in \mathfrak{p}\otimes\mathfrak{p}$ with respect to the basis $x_i \otimes x_j$ and define the form:

$$F^q_{\mathfrak{g}}(\xi,\eta) = (A/2q)\cdot\sum_{ij}\xi_{ij}\cdot\eta_{ij} + \sum_{ijkl} R_{ijkl}\eta_{il}\eta_{jk}$$

with A as above.

Let

$$m(\mathfrak{g}) = \max(\{0\}\cup\{q \mid F^q_{\mathfrak{g}} > 0\}).$$

(2.16) Theorem (Matsushima's Vanishing Theorem)**.** Let (π, V) be a unitary $(\mathfrak{g},\mathfrak{k})$-module on which the Casimir element acts by a scalar multiple of the identity and such that $V^{\mathfrak{g}} = 0$. Then $H^q(\mathfrak{g},\mathfrak{k};V) = 0$ for $q \leq m(\mathfrak{g})$.

Proof. If C acts non-trivially on V, then $H^q(\mathfrak{g},\mathfrak{k};V) = 0$ for all the q's by (2.2). From now on assume $\pi(C) = 0$.

We shall prove that if there exists $q \leq m(\mathfrak{g})$ such that $H^q(\mathfrak{g},\mathfrak{k};V) \neq 0$, then $V^{\mathfrak{g}} \neq 0$. If $q = 0$ then this is clear. Let $q \geq 1$.

Since $\pi(C) = 0$, all cochains are closed, harmonic and $H^q(\mathfrak{g},\mathfrak{k};V) = C^q(\mathfrak{g},\mathfrak{k};V)$ by (2.2) so we have to show that if $\eta = \sum_I \eta_I \cdot \omega^I$ is a cochain, then $\eta_I \in V^{\mathfrak{g}}$, i.e. $x_i\eta_I = x_a\eta_I = 0$ for all i, a, I.

Since $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$,

$$0 = (C\eta_I,\eta_I) = \sum_a \|x_a\eta_I\|^2 - \sum_i \|x_i\eta_I\|^2$$

so it suffices to prove $x_i\eta_I = 0$ for $1 \leq i \leq m, I \subset I_m, |I| = q$. $I_m = \{1,2,\dots,m\}$.

This also shows that if $v \in V^{\mathfrak{k}}$, then $v \in V^{\mathfrak{g}}$.

Let

$$\phi(\eta) = \frac{(q-1)!}{2}\sum_{i,j,I}\|[x_i,x_j]\eta_I\|^2 = (2q)^{-1}\cdot\sum_{i,j,i_1,\dots,j_q}\|[x_i,x_j]\eta_{j_1\dots j_q}\|^2.$$

we shall transform $\Phi(\eta)$ in two ways.

First, using $[x_i,x_j] = \sum c^a_{ij}x_a$ and $L(x_a,x_b) = -\sum_{ij} c^a_{ij}\cdot c^b_{ij}$ we can write

$$\Phi(\eta) = -\frac{(q-1)!}{2}\sum_{a,b,i,j,I} c^a_{ij}\cdot c^b_{ij}(x_a\eta_I,x_b\eta_I) = -\frac{(q-1)!}{2}\sum_{a,b,I} L(x_a,x_b)\cdot(x_a\eta_I,x_b\eta_I).$$

Since the x_a's are orthogonal with respect to L, the sum is over $a = b$ and, by the definition of A, we have

$$\Phi(\eta) \geq \frac{A\cdot(q-1)!}{2}\sum_{a,I}\|x_a\eta_I\|^2.$$

If we use the formula for $[x_i,x_j]$ on one term in each of the scalar products in our definition of $\Phi(\eta)$, we get

$$\Phi(\eta) = (2q)^{-1} \sum_{i,j,a,j_1,\dots,j_q} c_{ij}^a (x_a \cdot \eta_{j_1 \dots j_q}, [x_i, x_j] \cdot \eta_{j_1 \dots j_q})$$

and since c_{ij}^a and $[x_i, x_j]$ are antisymmetric in i, j, this gives

$$\Phi(\eta) = (q)^{-1} \sum_{i,j,a,j_1,\dots,j_q} c_{ij}^a (x_a \cdot \eta_{j_1 \dots j_q}, x_i \cdot x_j \cdot \eta_{j_1 \dots j_q})$$

By assumpton, $\eta \in C^q(\mathfrak{g}, \mathfrak{k}; V)$ so

$$\begin{aligned} x_a \cdot \eta_{j_1 \cdots j_q} &= \sum_u \eta(x_{j_1}, \dots, [x_a, x_{j_u}], \dots, x_{j_q}) \\ &= \sum_{a,k,u} c_{a,k_u}^k \eta(x_{j_1}, \dots, x_k, \dots, x_{j_q}) \\ &= \sum_{a,k,u} (-1)^{u-1} \cdot c_{a,j_u}^k \cdot \eta(x_{j_1}, x_{j_1}, \dots, \hat{x_{j_u}}, \dots x_{j_q}) \end{aligned}$$

so using $c_{ij}^a = c_{aj}^i$,

$$q \cdot \Phi(\eta) = \sum_{i,j,k,u,j_1,\dots,j_q} (-1)^{u-1} (\sum_a c_{ij}^a \cdot c_{kj_u}^a)(\eta_{k,j_1,\dots,j_u,\dots,j_q} x_i \cdot x_j \cdot \eta_{j_1 \dots j_q}).$$

Since (π, V) is unitary, we have

$$(\eta_{k,j_1,\dots,\hat{j_u},\dots,j_q} x_i \cdot x_j \cdot \eta_{j_1 \dots j_q}) = -(x_i \eta_{k,j_1,\dots,\hat{j_u},\dots,j_q} x_i \cdot x_j \cdot \eta_{j_1 \dots j_q})$$

and using $R_{ijkl} = B([[x_l, x_k], x_j], x_i) = B([x_l, x_k], [x_j, x_i])$ we get

$$\begin{aligned} q\Phi(\eta) &= \sum_{i,j,k,u,j_1,\dots,j_q} R_{ijkj_u} (x_i \cdot \eta_{k,j_1,\dots,\hat{j_u},\dots,j_q}, x_i \cdot \eta_{j_1 \dots \hat{j_u},\dots,j_q}) \\ &= q \sum_{i,j,k,l,j_2,\dots,j_q} R_{ijkl} (x_i \eta_{k,j_2,\dots,j_q} x_j \eta_{l,j_2,\dots,j_q}) \\ &= - \sum_{i,j,k,l,j_2,\dots,j_q} R_{ijkl} (x_i \eta_{k,j_2,\dots,j_q} x_j \eta_{l,j_2,\dots,j_q}) \end{aligned}$$

using that R_{ijkl} is antisymmetric in the last two indices (using relation to B).

Combining our two $\Phi(\eta)$ we get

$$\sum_{j_2,\dots,j_q} \left\{ \frac{A}{2q} \sum_{i,j} \|x_j \eta_{i,j_2,\dots,j_q}\|^2 + \sum_{i,j,k,l} R_{ijkl} (x_i \eta_{k,j_2,\dots,j_q}, x_j \eta_{l,j_2,\dots,j_q}) \right\} \leq 0.$$

On $\mathfrak{p} \otimes \mathfrak{p} \otimes V$ we consider the tensor product $F^q_{q,V}$ of $F^q_{\mathfrak{g}}$ and the given scalar product on V. It is positive non-degenerate since $q \leq m(\mathfrak{g})$ and the above inequality can be written as

$$\sum_j F^q_{\mathfrak{g},V}(\{x_j \cdot \eta_{i \cup J}) \leq 0$$

where J runs through the subsets of I_m with $q-1$ elements.

Since $F^q_{\mathfrak{g},V}$ is positive non-degenerate, we get

$$x_j \cdot \eta_{i \cup J} = 0$$

for all $1 \leq i, j \leq m$, $J \subset I_m$, $|J| = q-1$, which is the same as

$$x_i \eta_I = 0$$

for all $1 \leq i \leq m, I \subset I_m, |I| = q$. □

(2.17) Corollary. If (σ, H) is a nontrivial irreducible admissible unitary $(\mathfrak{g}, \mathfrak{k})$-module, then $H^q(\mathfrak{g}, \mathfrak{k}; H) = 0$ for $q \leq m(\mathfrak{g})$.

3 Matsushima's formula

Let $\Gamma \leq G$ be a discrete subgroup of a Lie group G with finitely many connected components and maximal compact K. Let (ρ, E) be a finite dimensional irreduible complex representation of G and (σ, H) a unitary $(\mathfrak{g}, K)$-module. Let $V = H \otimes E$. We wish to prove Matushima's formula, that $H^*(\Gamma, E) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma, E) H^*(\mathfrak{g}, K; H_{\pi,0})$.

We first prove that $H^q(\Gamma, V) = H^q(\Gamma \setminus X, \tilde{V}) = H^q(\mathfrak{g}, K, C^\infty(\Gamma \setminus G) \otimes V)$.

G acts on $L^2(\Gamma \setminus G, V)$ and it decomposes as a direct sum of irreducible representations with finite multiplicities $\widehat{\bigoplus}_\pi m(\pi, \Gamma) \cdot H_\pi$. Moreover $\mathcal{C}^\infty(\Gamma \setminus G) = (L^2(\Gamma \setminus G))^\infty$ so $H^*(\mathfrak{g}, K; \mathcal{C}^\infty(\Gamma \setminus G) \otimes V) = H^*(\mathfrak{g}, K; \left(\widehat{\bigoplus}_\pi m(\pi, \Gamma) \cdot H_\pi\right)^\infty \otimes V)$.

For $S \subset \hat{G}$ be a finite set of representations, we can decompose $H^*(\Gamma, V) = \bigoplus_{\pi \in S} m(\pi, \Gamma) H^*(\mathfrak{g}, K; H_\pi \otimes V) \oplus^* (\mathfrak{g}, K; \left(\widehat{\bigoplus_{\pi \notin S} m(\pi, \Gamma) \cdot H_\pi}\right)^\infty \otimes V)$.

By compactness, $H^*(\Gamma, V)$ is finite dimensional, so for large enough S, $H^*(\mathfrak{g}, K; H_\pi \otimes V) = 0$ for all $\pi \notin S$. Then it is enough to prove that if each $(\mathfrak{g}, K)$-cohomology of a countable collection of irreducible unitary representations of G vanishes, then the cohomology of its direct sum vanishes also.

(3.1) Notation. Let M be a smooth manifold, $L = \mathbb{R}$ or $\mathbb{C}$, E a finite dimensional vector space over L. Let $T(M)_m$ denote the tangent space at $m \in M$ and $C^\infty(M; E)$ the space of smooth functions with values in E.

Let $A_q(M; L)$ be the space of smooth vector fields on M and $A_1(M; L) = A_1(M) \otimes_{\mathbb{R}} L$.

Let $A^q(M; E)$ be the space of smooth E-valued differential q-forms on M. If $\omega \in A^q(M; E)$, it associated to each $m \in M$, an element of $\mathrm{Hom}(\bigwedge^q T(M)_m, E)$. Let the value of ω on $y = (y_1, \ldots, y_q)$ at m be denoted $\omega(m; y)$. We can define a differential $d : A^q(M; E) \to A^{q+1}(M : E)$ in the same way as for relative $(\mathfrak{g}, \mathfrak{k})$-cohomology.

(3.2) Remark. Let $\tilde{E}$ be the local system of coefficients on M associated to a representation on E of the fundamental group (there is a 1-1 correspondence between representations of the fundamental group and local systems). Since the transition functions on $\tilde{E}$ are locally constant, we can still define $A^q(M;\tilde{E})$, and exterior differentiation d in the same way.

Suppose now that G^0 has finitely many connected components.
Let M be a compact subgroup of G.

(3.3) Theorem. There is a canonical isomorphism

$$H^*(\Gamma;E) \cong H^*(A(X;E)^\Gamma).$$

Proof. We will prove under assumption that there is $\Gamma' \le \Gamma$ a torsion free subgroup of finite index.

Suppose that Γ acts freely. Then $\Gamma \setminus X$ is a smooth manifold and, since X is contractible, it is also an Eilenberg-MacLand space $K(\Gamma,1)$. That is, it is a connected topological space with homotopy group $\pi(\Gamma \setminus X) = \Gamma$. Then

$$H^*(\Gamma;E) \cong H^*(\Gamma \setminus X;\tilde{E}).$$

Let $\pi : X \to \Gamma \setminus X$ be the projection. It is immediate that $\omega \mapsto \omega \circ \pi$ defines an isomorphism $A(\Gamma \setminus X;\tilde{E}) \to A(X;E)^\Gamma$ and then applying de Rhams's theorem gives the result.

Assume instead that there is $\Gamma' \le \Gamma$ a torsion free subgroup of finite index. Then Γ/Γ' acts on $H^*(\Gamma';E)$ and, from the Hochschild-Serre spectral sequence, $H^*(\Gamma;E) = (H^*(\Gamma';E))^{\Gamma/\Gamma'}$. Also, $A(X;E)^\Gamma = (A(X;E)^{\Gamma'})^{\Gamma/\Gamma'}$ and since taking invariants under a finite group is an exact functor in characteristic zero, it follows that $H^*(A(X;E)^\Gamma) = H^*(A(X;E)^{\Gamma'})^{\Gamma/\Gamma'}$ and we can reduce to the first case. □

(3.4) Lemma. If E comes from a representation of G, then there is a canonical isomorphism

$$H^*(\Gamma \setminus G;\tilde{E}) \cong H^*(\mathfrak{g};C^\infty(\Gamma \setminus G;L) \otimes E).$$

Proof. For $g \in G$, left translaton by g^{-1} gives a canonical isomorphism $T(G)_g \cong T(G)_1 = \mathfrak{g}$, so an identification

$$\iota : A^q(G;E) \cong \mathrm{Hom}(\bigwedge^q(\mathfrak{g}), C^\infty(G;E)) = C^q(\mathfrak{g}, C^\infty(G;E)).$$

For some $\omega \in A^q(G;E)^\Gamma$ and $y \in \bigwedge^q \mathfrak{g}$, $\omega(\gamma \cdot d, u) = \rho(y) \cdot \omega(g,y)$, so ω can be identified with an element of $\mathrm{Hom}(\bigwedge^q \mathfrak{g}, I^\infty(E))$. This gives an isomorphism

$$\iota : A^q(G;E)^\Gamma \cong C^q(\mathfrak{g};I^\infty).$$

These ι commute with the differentials so give rise to

$$\iota^* : H^*(A(G;E)^\Gamma) \cong H^*(\mathfrak{g};I^\infty). \tag{8}$$

Now for $\tilde{E}$ a local system, $A(G;E)^\Gamma \cong A(\Gamma \setminus G;\tilde{E})$, so the left hand side of (8) is the cohomology of $\Gamma \setminus G$ with coefficients in $\tilde{E}$).

If E is the restriction of a representation of G, then $C^\infty(G;E) \xrightarrow{\omega \mapsto \omega^0} C^\infty(G;E)$ where $\omega^0(g) = \rho(g)^{-1}\omega(g)$ induces an isomorphism of G modules $I^\infty(R) \xrightarrow{\sim} C^\infty(\Gamma \setminus G;L) \otimes_L E$ (here the right hand side action is given by the tensor product of the right regular representation by ρ).

So there is an isomorphism

$$A(G;E)^\Gamma \xrightarrow{\sim} C^*(\mathfrak{g};C^\infty(\Gamma \setminus G;L) \otimes E)$$

and so

$$H^*(\Gamma \setminus G;\tilde{E}) \xrightarrow{\sim} H^*(\mathfrak{g};C^\infty(\Gamma \setminus G;L) \otimes E).$$

□

(3.5) Lemma. There is an isomorphism

$$H^*(\Gamma;E) \cong H^*(\mathfrak{g},K;I^\infty(E)).$$

Proof. Let $\pi : G \to X = G/K$ be the canonical projection. Consider ${}^t\pi : A(X;E)^\Gamma \xrightarrow{\omega \mapsto \omega \circ \pi} A(G;E)^\Gamma$. ${}^t\pi$ commutes with graded left translations, so it does indeed land in $A(G;E)^\Gamma$. Let A_0 be its image.

π is constant on the left K-cosets so A_0 consists of elements of $A(G;E)^\Gamma$ right invariant under K and annihilated by interior products i_x for $x \in \mathfrak{k}$.

Therefore $\iota \circ \pi$ induces an isomorphism

$$A(X;E)^\Gamma \cong C^*(\mathfrak{g},K;I^\infty(E))$$

and the result follows. □

Assume now that E is a unitary Γ-module with scalar product $(\ ,\ )_E$ and $L = \mathbb{C}$.

If $u,v \in I^\infty(E)$, then $(u(\gamma \cdot x), v(\gamma \cdot x))_E = (u(x),v(x))_E := f_{u,v}(x,y)$ for $x \in G, \gamma \in \Gamma$. So $f_{u,v}$ is a function on $\Gamma \setminus G$, so we can define a global scalar product (u,v) by integrating it over $\Gamma \setminus G$. Completing $I^\infty(E)$ under this gives the space $I_2(E)$ of square integrable cross-sections[1] of the bundle $G \times_\Gamma E \to \Gamma \setminus G$ (where $G \times_\Gamma E = G \times E/\sim$ where $(g,e) \sim (\gamma \cdot g, \rho(\gamma)e)$ for $\gamma \in \Gamma$).

$I_2(E)$ is a unitary G-module with respect to right translations.

(3.6) Remark. $(I_2(E))^\infty = I^\infty(E)$ topologically. Fill this in

[1]Explicitly, to define the local system $\tilde{V}$ attached to V, define the sheaf over U to be $\mathcal{G}(U) = \{k : \tilde{U} \to V \mid f(\gamma \cdot x) = \gamma \cdot f(x) \,\forall \gamma \in \Gamma\}$ where $\tilde{U}$ is the lift of U to G. Then for any $x \in X$ and U connected open containing it, the map $\mathcal{G}(U) \xrightarrow{f \mapsto f(x)} V$ is an isomorphism. The same holds for any open connected $U' \subset U$, so $\mathcal{G}|_U \cong \underline{E}_U$ and so it is locally constant, i.e. a local system. Conversely, given a locally constant sheaf, any path $p : [0,1] \to X$ determines a bijection $\mathcal{F}_{p(0)} \xrightarrow{\sim} \mathcal{F}_{p(1)}$ and this allows us to define an invertible linear map $\rho(p) : \mathcal{F}_{p(0)} \to \mathcal{F}_{\gamma(1)}$. We can show ρ is homotopy invariant so $\rho : \pi_1(X) \to \mathrm{GL}(\mathcal{F})_{x_0}$. These correspondences in fact give us an equivalence of categories between local systems and representations of Γ.

(3.7) Fact (Gelfand and Piatetski-Shapiro). $I_2(E)$ decomposes into a discrete Hilbert direct sum of irreducible G-modules of finite multiplicities

$$I_2(E) = \widetilde{\bigoplus}_{\pi\in\hat{G}} m(\pi,\Gamma,E)H_\pi$$

where $m(\pi,\Gamma,E)\in\mathbb{N}$.

This imples that

$$I^\infty(E) = \left(\widetilde{\bigoplus}_{\pi\in\hat{G}} m(\pi,\Gamma,E)H_\pi\right)^\infty.$$

(3.8) Lemma. Let T be a countable set of irreducible unitary representations H_π of G, and V the Hilbert direct sum of the H_π's. Assume the $H^*(\mathfrak{g},K;H_\pi^\infty) = (\pi,H_\pi) = 0$ for all $\pi\in T$ and that $H^*(\mathfrak{g},K;V)$ is finite dimensional. Then $H^*(\mathfrak{g},K;V^\infty) = 0$.

Proof. Let $C^*(V) = C^*(\mathfrak{g},K;V^\infty)$, a topological direct sum of finitely many copies of V^∞. We can show that $d : C^{q-1}(V^\infty)\to C^q(V^\infty)$ is continuous, so $Z^q = C^q(V^\infty)\cap\ker d$ is closed. There is an exact sequence

$$0\to dC^{q-1}(V^\infty)\to Z^q\to H^q(C^*(V^\infty))\to 0.$$

Since $H^*(C^*(V^\infty))$ is finite dimensional, $dC^{q-1}(V^\infty)$ has finite codimension in Z^q so has closed complement. As they are Fréchet spaces, $dC^{q-1}(V^\infty)$ is closed.

If $S\subset T$ is finite, let $\mathrm{pr}_S : V^\infty : \widetilde{\bigoplus}_{\pi\in T}H_\pi^\infty \to \widetilde{\bigoplus}_{\pi\in S}H_\pi^\infty$ be the projection with kernel $\left(\widetilde{\bigoplus}_{\pi\in T-S}H_\pi\right)^\infty$. This defines another projection

$$\mathrm{pr}_S : C^*(V^\infty)\to\bigoplus_{\pi\in S}C_\pi^*$$

with kernel $C^*\left(\left(\widetilde{\bigoplus}_{\pi\in T}H_\pi\right)^\infty\right)$. It then follows from the definition of the topology on $C^*(V^\infty)$ that $x\in C^*(V^\infty)$ is $\lim_{S\to T}\mathrm{pr}_S x$.

Let $z\in Z^q$. By assumption, $\mathrm{pr}_S z$ is a coboundary for every finite S. Since z is the limit of $\mathrm{pr}_S z$, it is in the closure of $dC^{q-1}(V^\infty)$ which is a closed space, so $z\in dC^{q-1}(V^\infty$ and we are done. □

(3.9) Theorem. We have

$$H^*(\Gamma,E) = \bigoplus_{\pi\in\hat{G}} m(\pi,\Gamma,E)H^*(\mathfrak{g},K;H_{\pi,0}).$$

This sum is finite and may be restricted to $\pi\in\hat{G}$ with trivial infinitesimal and central characters.

Proof. We have

$$H^*(\Gamma;E) = H^*(\mathfrak{g},K;\left(\widetilde{\bigoplus}_{\pi\in\hat{G}} m(\pi,\Gamma,E)H_\pi\right)^\infty).$$

For $\pi \in \hat{G}$, $q \in \mathbb{N}$, let

$$C_\pi^q = C^q(\mathfrak{g}, K; m(\pi, \Gamma, E) H_\pi^\infty)$$
$$C_\pi^* = \bigoplus_q C_\pi^q.$$

Let $S \subset \hat{G}$ be finite and put

$$C_{S'}^* = C^* \left(\mathfrak{g}, K; \left(\widehat{\bigoplus_{\pi \in \hat{G} - S}} m(\pi, \Gamma, E) H_\pi \right)^\infty \right).$$

Then $C^*(\mathfrak{g}, K; I^\infty(E)) = C_S^* \oplus C_{S'}^*$, and hence

$$H^*(\Gamma, E) = \bigoplus_{\pi \in S} m(\pi, \Gamma, E) H^*(\mathfrak{g}, K; H_\pi^\infty) \oplus H^*(C_{S'}^*)$$

The space $\Gamma \setminus X$ is compact and locally contractable (note sure why need locally contractible), and $H^*(\Gamma; E)$ is the cohomology of $\Gamma \setminus X$ with coefficients in a $\tilde{E}$, $\dim H^*(\Gamma; E)$ is finite dimensional.

Then there exists a finite set $S \subset \hat{G}$ such that $H^*(C_\pi^*) = 0$ for $\pi \notin S$ with $m(\pi, \Gamma, E) \neq 0$. Given such S, we now must show $H^*(C_{S'}^*) = 0$ and we will be done.

$H^*(C_S^*)$ is finite dimensional as it is a direct summand of $H^*(\Gamma, E)$, so it is enough to prove that if each $(\mathfrak{g}, K)$-cohomology of a countable collection of irreducible unitary representations of G vanishes, then the cohomology of its direct sum vanishes also. So the result follows from the previous lemma.

Finally, by (2.11) we can restrict to $\pi \in \hat{G}$ with trivial infinitesimal and central characters. □

Now using (2.4):

(3.10) Corollary. Suppose G is reductive. Then

$$H^q(\Gamma, E) = \bigoplus_{\pi \in \hat{G}, \chi_\pi = \chi_0, \omega_\pi = \omega_0} m(\pi, \Gamma, E) \operatorname{Hom}(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), H_{\pi,0})$$

In the case that Γ is also a cocompact subgroup and E a finite dimensional G-module, we also have direct sum decompositions with finite multiplicities

$$L^2(\Gamma \setminus G) = \widetilde{\bigoplus}_{\pi \in \hat{G}} m(\pi, \Gamma) H_\pi$$

and

$$C^\infty(\Gamma \setminus G) = (L^2(\Gamma \setminus G))^\infty = \left(\widetilde{\bigoplus}_{\pi \in \hat{G}} m(\pi, \Gamma) H_\pi \right)^\infty$$

and moreover,

$$I^\infty(E) \cong \left(\widetilde{\bigoplus}_{\pi \in \hat{G}} m(\pi, \Gamma) H_\pi \right)^\infty \otimes E.$$

So then it follows that

(3.11) Theorem.

$$H^*(\Gamma; E) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, K; H_{\pi,0} \otimes E).$$

So now, we see that it is of interest to study the cohomology groups $H^k(\mathfrak{g}, K; H_\pi \otimes V)$. In the next section, we show that they vanish outside of a certain range and give a formula for their dimension.

(3.12) Remark. Using Matsushima's formula, we can split according to the isomorphism class of π. Vogan and Zuckerman ([6]) classified the representations with non-vanishing $(\mathfrak{g}, K)$-cohomology and they are indexed by θ-stable parabolic subgroups so we can write

$$H^q(\Gamma, \mathbb{C}) = \bigoplus_{\mathfrak{p}} H^q(\Gamma, \mathbb{C})_{\mathfrak{p}}.$$

Then the tempered cohomology $H^q(\Gamma, \mathbb{C})_{\text{temp}}$ described in (1.33) corresponds to minimal $\mathfrak{p}$ and the invariant to maximal $\mathfrak{p}$.

4 Cohomology of induced representations

We will now prove a theorem giving the dimension of $H^q(\mathfrak{g}, K; F \otimes I)$ for G connected and reductive, F a finite dimensional representation, and I the K-finite vectors of an induced representation of a parabolic subgroup of G. This allows us to completely determine $H^q(\mathfrak{g}, K; F \otimes I)$ in the case that V is tempered.

We start by defining induced representations and prove a version of Shapiro's lemma:

$$H^q(\mathfrak{g}, K; I \otimes F_\lambda) = H^q(\mathfrak{p}, K_P; H_{\sigma,\nu} \otimes F_\lambda)$$

where $H_{\sigma,\nu} = H_\sigma \otimes \mathbb{C}_{\rho_P + \nu}$.

The existence of a Hochschild-Serre spectral sequence

$$E_2^{p,q} := H^p(\mathfrak{m}, K_P; H^q(\mathfrak{n}; F_\lambda) \otimes H_{\sigma,V}) \Rightarrow H^{p+q}(\mathfrak{p}, K_P; H_{\sigma,\nu} \otimes F_\lambda)$$

reduces the problem to understanding the $(\mathfrak{m}, K_P)$-representations $H^q(\mathfrak{n}; F_\lambda)$. However, a theorem of B.Konstant allows us to replace these by $\bigoplus_s L_s$, where $L_s = E_{s(\rho+\lambda)-\rho}$ are irreducible representations of M with highest weight $s(\rho + \lambda) - \rho$ and the sum is over representatives of the quotient of Weyl groups $W_M \setminus W_G$ of length $l(s) = q$.

Using this, we then prove that

$$H^{q+l(s)}(\mathfrak{g}, K; I \otimes F_\lambda) = (H^*({}^0\mathfrak{m}, K_P; H_\sigma \otimes E_{(s(\rho+\lambda)-\rho)}) \otimes \bigwedge^* \mathfrak{a}_{\mathbb{C}}^*)^q.$$

Now we already proven that the righthand side is concentrated in degree $q({}^0M)$ and is 1-dimensional. Also $\dim(\mathfrak{a}_{\mathbb{C}}^*) = l_0$, so

$$H^{q+l(s)}(\mathfrak{g}, K; I \otimes F_\lambda) = \bigwedge^{q-q({}^0M)} \mathfrak{a}_{\mathbb{C}}^*.$$

It is then left to prove that $l(s) = \frac{\dim N}{2}$ and $q_0(G) = q({}^0M) + \frac{\dim N}{2}$, which shows that the dimension only vanishes for $q \notin [q_0, q_0 + l_0]$.

4.1 Induced representations and their K-finite vectors

In this section we show how to, given a differentiable admissible representation of M into a Hilbert space, induce the underlying $(\mathfrak{m}, K_P)$-module H_0 of K_P-finite vectors to a $(\mathfrak{g}, K)$-module $I(H_0)$. We will then prove a version of Shapiro's lemma, that $H^*(\mathfrak{p}, K_P; H_0) \xrightarrow{\sim} H^*(\mathfrak{g}, K; I(H_0))$, by constructing an isomorphism between the corresponding complexes.

(4.1) Definition. Let R be a closed subgroup of a Lie group Q and let (π, V) be a differentiable R-module. Then we can define the ***representation of*** V ***induced to*** Q, a differentiable representation of Q

$$\mathrm{Ind}_R^Q(V) = \{f \in C^\infty(Q; V) \mid f(r \cdot q) = \pi(r) \cdot f(q) \, \forall q \in Q, r \in R\}$$

where Q acts by right translations $(q_1 \cdot f)(q_2) = f(q_2 q_1)$.

(4.2) Remark. If (τ, U) is a finite dimensional continuous representation of Q, then there is a canonical isomorphism

$$\zeta : \mathrm{Ind}_R^Q(V \otimes U) \xrightarrow{\sim} (\mathrm{Ind}_R^Q(V)) \otimes U$$
$$\zeta(f)(q) = \tau(q)^{-1} f(q).$$

Let (P, A) be a semi-standard parabolic p-pair in G, $P = MN$ the standard Levi decomposition, $K_P = K \cap P$ a maximal compact sugbgroup of P contained in M.

Let (σ, H) be a differentiable admissible representation of M into a Hilbert space H and let H_0 be the $(\mathfrak{m}, K_P)$-module of K_P-finite vectors in H. We can view H as a P-module where N acts trivially. Assume that σ has a central character.

Write

$$I(\sigma) = \mathrm{Ind}_P^G(H).$$

The idea of this section is to give a version of Shapiro's lemma (a generalisation of Frobenius reciprocity) for $(\mathfrak{g}, K)$-modules.

Let V be an $(\mathfrak{m}, K_P)$-module. We wish to induce it to a $(\mathfrak{g}, K)$-module. Let

$$U_0 := \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), V) = \{f : U(\mathfrak{g}) \to V \mid f(pg) = pf(g) \, \forall p \in P, g \in G\}$$

is a $U(\mathfrak{g})$-module under right multiplication.

Further, let

$$U_1 := \{f \in U_0 \mid f \text{ is } \mathfrak{k}\text{-finite and } U(\mathfrak{k})f \text{ is a completely reducible } \mathfrak{k}\text{-module}\}.$$

Let $\tilde{K}$ denote the simply connected covering group of K. Then since U_1 is a direct sum of irreducible representations of $\mathfrak{k}$, and as $\tilde{K}$ is simply connected, we can uniquely lift this to a $\tilde{K}$-module structure on U_1 such that the differential of the representation agrees with the given action of $\mathfrak{k}$.

Let $Z = \ker p$ where $p : \tilde{K} \to K$ is the covering morphism. Then $Z \subset \tilde{K}_P = \tilde{K} \cap p^{-1}(P)$ as $1 \in P$.

Let

$$U := \{f \in U_1 \mid (mf)(x) = p(m) \cdot f(\mathrm{Ad}(m)^{-1} x) \, \forall m \in \tilde{K}_P, x \in \mathfrak{g}\}.$$

We wish to show that U is a $(\mathfrak{g}, K)$-module. If $x, y \in U(\mathfrak{g})$, $m \in \tilde{K}_P$ and $f \in U$, then

$$\begin{aligned} m(yf)(x) &= (\mathrm{Ad}(m)y)(mf)(x) \\ &= mf(x\,\mathrm{Ad}(m)y) \\ &= p(m)\cdot f(\mathrm{Ad}(m)^{-1}(x\,\mathrm{Ad}(m)y)) \quad \text{as } f \in U \\ &= p(m)\cdot f((\mathrm{Ad}(m)^{-1}x)y) \quad \text{as } m^{-1}(xmym^{-1})m = (m^{-1}xm)y \\ &= p(m)\cdot yf(\mathrm{Ad}(m)^{-1}x) \end{aligned}$$

so $yf \in U$. So U is $\mathfrak{g}$-invariant. U is also a $\tilde{K}$-invariant subspace of U_1.

Z acts by the identity on U so the action of $\tilde{K}$ on U pushes down to K and hence U is a $(\mathfrak{g}, K)$-module.

(4.3) Definition. We will use the notation $\mathrm{Ind}_{(\mathfrak{p},K_P)}^{(\mathfrak{g},K)}(V)$ for U constructed as above and call it the $(\mathfrak{g}, K)$***-module parabolically induced from*** V. If $\mathfrak{g}, K, P$ are understood, we use the notation

$$I(V) := \mathrm{Ind}_{(\mathfrak{p},K_P)}^{(\mathfrak{g},K)}(V) = U.$$

(4.4) Proposition. Let (σ, H) be a differentiable admissible representation of M and let H_0 be the underlying $(\mathfrak{m}, K_P)$-module.

Let

$$\begin{aligned} T : I(H) &\to \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), H_0) \\ Tf(x) &= xf(1) \quad \text{for } f \in I(H), x \in U(\mathfrak{g}). \end{aligned}$$

If I_0 is the space of K-finite vectors in $I(H)$, then $T(I_0) = I(H_0)$ and $T : I_0 \xrightarrow{\sim} I(H_0)$ defines an isomorphism of $(\mathfrak{g}, K)$-modules.

Proof. Let $x, y \in U(\mathfrak{g})$, $f \in I(H)$. Then

$$\begin{aligned} y \cdot Tf(x) &= Tf(xy) \\ &= xyf(1) \\ &= yf(x) \\ &= T(yf)(x) \end{aligned}$$

so T commutes with $U(\mathfrak{g})$. Let $p \in \mathfrak{p}$, $x \in U(\mathfrak{g})$. Then

$$\begin{aligned} (Tf)(px) &= pxf(1) \\ &= \frac{d}{dt} xf(\exp(tp)|_{t=0} \\ &= \frac{d}{dt}\sigma(\exp(tp)) \cdot xf(1)|_{t=0} \\ &= \sigma(p) \cdot Tf(x). \end{aligned}$$

This extends to $p \in U(\mathfrak{p})$ and shows that $\mathrm{Im}\, T \subset U_0$, so it is in U_1 also. Since it is a$\mathfrak{g}$-morphism, T is a $(\mathfrak{g}, K)$-morphism $I_0 \to U_1$.

We want to show now that $\operatorname{Im} T \subset U$. Let $f \in I_0$, $x \in \mathfrak{g}$, $m \in K_P$. Then

$$\begin{aligned}(m \cdot Tf)(x) &= T)mf)(x) \\ &= x \cdot (mf)(1) \\ &= \frac{d}{dt}(mf)(\exp(tx))|_{t=0}.\end{aligned}$$

For $y \in G$, we have

$$\begin{aligned}(mf)(y) &= f(ym) \\ &= f(mm^{-1}ym) \\ &= \sigma(m) \cdot f(m^{-1}ym)\end{aligned}$$

so for $x \in \mathfrak{g}$,

$$\begin{aligned}\frac{d}{dr}(mf)(\exp(tx))|_{t=0} &= \frac{d}{dt}\sigma(m) \cdot f(\operatorname{Ad} m^{-1}(\exp(tx))|_{t=0} \\ &= (\sigma(m) \cdot (\operatorname{Ad} m^{-1})(x) \cdot f)(1).\end{aligned}$$

So for $m \in K_M, x \in \mathfrak{g}, f \in I_0$,

$$mTf(x) = \sigma(m) \cdot f(\operatorname{Ad} m^{-1}(x)).$$

This extends to $x \in U(\mathfrak{g})$ so $Tf \in U$.

If $Tf = 0$, then $xf(1) = 0$ for all $x \in U(\mathfrak{g})$. But f is K-finite by the definition of I_0 and they are $\mathfrak{Z}(\mathfrak{g})$-finite so are analytic, so this implies that $f = 0$ so T is injective.

To show surjectivity, we wish to construct an inverse to T. Let $f \in U$ and for $k \in K$ define

$$Sf(k) := (kf)(1).$$

We have

$$\begin{aligned}Sf(mk) &= (mkf)(1) \\ &= \sigma(m)(kf)(\operatorname{Ad}(m)^{-1}(1)) \\ &= \sigma(m)(kf)(1) \\ &= \sigma(m) \cdot Sf(k).\end{aligned}$$

Therefore we may extend Sf to $G = PK$ by defining

$$Sf(p \cdot k) := \sigma(p) \cdot Sf(k)$$

for $p \in P, k \in K$.

It is clear that Sf is K-finite so $\operatorname{Im} S \subset I_0$. Further by the definition of S, for $y \in U(\mathfrak{k})$, $f \in U$,

$$y \cdot Sf(1) = f(y) \tag{9}$$

since the K-action of U is obtained by integrating right translations.

To prove surjectivity, it suffices to prove that $T \cdot S = \mathrm{Id}$.
For $x \in U(\mathfrak{p})$, $y \in U(\mathfrak{k})$, by (9)

$$\begin{aligned} TSf(xy) &= \sigma(x)TSf(y) \\ &= \sigma(x) \cdot y(Sf)(1) \\ &= \sigma(x)f(y) \\ &= f(xy) \end{aligned}$$

which proves the result as $U(\mathfrak{g}) = U(\mathfrak{p}) \cdot U(\mathfrak{k})$. □

(4.5) Proposition. Let H_0 be as in the previous theorem as let V be a $(\mathfrak{g}, K)$-module. There are canonical isomorphisms

1. $\mathrm{Hom}_{\mathfrak{p},K_P}(V, H_0) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g},K}(V, I(H_0))$;
2. $H^*(\mathfrak{p}, K_P; H_0) \xrightarrow{\sim} H^*(\mathfrak{g}, K; I(H_0))$.

Proof. Let $U = I(H_0)$.

1. Let $f \in \mathrm{Hom}_{\mathfrak{g},K}(V, U)$. Let $Tf : V \to U$ be defined by $Tf(v) = f(v)(1)$. Then given $g \in \mathrm{Hom}_{\mathfrak{p},K_P}(V, H_0)$ define $Sg : V \to \mathrm{Hom}(U(\mathfrak{g}), H_0)$ by

$$Sg(v)(r) = g(r \cdot v)$$

for $v \in V$, $r \in U(\mathfrak{g})$.

We can check as before that $TS = \mathrm{Id}$, $\mathrm{Im}\, S \subset \mathrm{Hom}_{\mathfrak{g},K}(V, U)$, $\mathrm{Im}\, T \subset \mathrm{Hom}_{\mathfrak{p},K_P}(V, H_0)$ and that T is injective which proves the (1).

2. The left hand size is the cohomology of the complex C^* where

$$C^i = \mathrm{Hom}_{\mathfrak{g},K}(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \bigwedge^i (\mathfrak{g}/\mathfrak{k}), U)$$

and the right hand side is the cohomology of the complex D^* for

$$D^i = \mathrm{Hom}_{\mathfrak{p},K_P}(U(\mathfrak{p}) \otimes_{U(\mathfrak{k}_P)} \bigwedge^i (\mathfrak{p}/\mathfrak{k}_P), H_0).$$

By (1), we have

$$C^i = \mathrm{Hom}_{\mathfrak{p},K_P}(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \bigwedge^i (\mathfrak{g}/\mathfrak{k}), H_0).$$

Not $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, so $\mathfrak{p}/\mathfrak{k}_P = \mathfrak{g}/\mathfrak{k}$, $U(\mathfrak{g}) = U(\mathfrak{p}) \cdot U(\mathfrak{k})$. This means that there exists a subspace Q of $U(\mathfrak{p}$ stable under the adjoint representation restricted to K_P, such that $U(\mathfrak{p}) = Q \otimes U(\mathfrak{k})$.We have vector space isomorphisms

$$\begin{aligned} U(\mathfrak{g}) &= Q \otimes U(\mathfrak{k}) \\ U(\mathfrak{p}) &= Q \otimes U(\mathfrak{k}_P). \end{aligned}$$

It follows that the natural map

$$U(\mathfrak{p}) \otimes_{U(\mathfrak{k}_P)} \bigwedge^{i} (\mathfrak{p}/\mathfrak{k}_P) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \bigwedge^{i} (\mathfrak{g}/\mathfrak{k})$$

induced by inclusions is an isomorphism of $(\mathfrak{p}, K_P)$-modules. This gives us an isomorphism of C^* onto D^*, so their cohomology groups are the same.

□

4.2 Cohomology with respect to principal series representations

Now fix a standard p-pair (P, A) of G with standard Levi decomposition $P = MN$.

Let (σ, H) be a differentiable admissible Fréchet ${}^0M$-module (i.e. ${}^0M$ is a complete Hausdorff metrizable locally convex $\mathbb{C}$-algebra and ${}^0M \times X \to X$ is continuous) with infinitesimal character χ_σ.

For $\mu \in \mathfrak{b}_c^*$, let E_μ denote the irreducible M_c-module with extreme weight μ.

Let $\nu \in \mathfrak{a}_c^*$.

Let $I = I_{P,\sigma,\nu}$ be as in (1.36) and notice that, where $\mathbb{C}_\mu$ denotes $\mathbb{C}$ acted on by A via $\mu \in \mathfrak{a}_c^*$,

$$\begin{aligned} I_{P,\sigma,\nu} &= \operatorname{Ind}_P^G(H_\sigma \otimes \mathbb{C}_{\rho_P+\nu}) \\ &= \{ f \in C^\infty(G; H_\sigma \otimes \mathbb{C}_{\rho_P+\nu}) \mid f(p \cdot g) = (\sigma \otimes (\rho_P + \nu)|_A)(p) \cdot f(g) \, \forall p \in P, g \in G \}. \end{aligned}$$

It is an admissible finitely generated Fréhet G-module with infinitesimal character $\chi_{\lambda_\sigma+\nu}$ if $\lambda_\sigma \in \mathfrak{b}_c^*$ is such that $\chi_\sigma = \chi_{\lambda_\sigma}$ (this follows as if (π, V) is an irreducible admissible representation with infinitesimal character χ_π of a linear reductive group of connected type, if V is finite dimensional with highest weight μ, then $\chi_\pi = \chi_{\mu+\rho}$).

(4.6) Theorem (Konstant's Theorem)**.** There is an isomorphism of ${}^0M$ modules

$$H^j(\mathfrak{n}_P, F_\lambda) = \bigoplus_{s \in W^P, l(s)=j} E_{s(\rho+\lambda)-\rho},$$

a multiplicity free decomposition as an M_c-module..

(4.7) Theorem. Let L be a compact Lie group such that $\mathfrak{l}$ contains an ideal isomorphic to $\mathfrak{k}$ and L with corresponding analytic subgroup K. Let $\mathcal{C}_{\mathfrak{g},\mathfrak{k},L}$ be the category of $(\mathfrak{g}, \mathfrak{k}, K)$-modules ($V$ is a $(\mathfrak{g}, K)$-module with an action of L such that V is locally finite and semisimmple and $x(u \cdot v) = x(u) \cdot x(v)$ for $x \in L, u \in U(\mathfrak{g}), v \in V$ and any finite dimensional K-stable subspace M of V is stable under $\mathfrak{k}$ and the differential coincides with the $\mathfrak{k}$-action) (page 19).

Let $\mathfrak{n}$ be an ideal in $\mathfrak{g}$ stable under L, $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{n}$, K_1 a closed normal subgroup of L with Lie algebra $\mathfrak{k}_1$, and $V \in \mathcal{C}_{\mathfrak{g},\mathfrak{k},L}$ (. Then there exists a spectral sequence which converges to $H^*(\mathfrak{g}, \mathfrak{k}; V)$ in which

$$E_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{n}, \mathfrak{k}/\mathfrak{k}_1; H^q(\mathfrak{n}, \mathfrak{k}_1; V))$$

and a spectral sequence which converges to $H^*(\mathfrak{g}, K; V)$ and in which

$$E_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{n}, L/K_1, H^q(\mathfrak{n}, K_1; V))$$

for $p, q \in \mathbb{Z}$.

We wish to prove the following:

(4.8) Theorem. Let $\mathfrak{h}_c^*$ be a dominant weight and F_λ a simple G_c-module with highest weight λ.

(i) If $H^*(\mathfrak{g}, K; I \otimes F_\lambda) \neq 0$, then there exists $s \in W^P$ such that

  (1) $s(\rho + \lambda)|_A + \nu = 0$,

  (2) $\chi_\sigma = \chi_{-s(\rho+\lambda)}|_{\mathfrak{b}_c}$.

  Such an s is unique.

(ii) If $s \in W^P$ satisfied (1) and (2), then, for every $q \in \mathbb{N}$, we have

$$H^{q+l(s)}(\mathfrak{g}, K; I \otimes F_\lambda) = (H^*({}^0\mathfrak{m}, K_P; H_\sigma \otimes E_{(s(\rho+\lambda)-\rho)}) \otimes \Lambda \mathfrak{a}_c^*)^q. \tag{3}$$

(4.9) Remark. Conditions (1) and (2) are equivalent to $-(\rho + \lambda) \in W(\lambda\sigma + \nu)$, and (1) implies that ν is real valued.

(4.10) Remark. In (3), $E_{s(\rho+\lambda)-\rho)}$ is viewed as a ${}^0M$-module by restriction and, since $M ={}^0 M \times A$ with A a commutative group, $E_{s(\rho+\lambda)-\rho}$ is an irreducible ${}^0M$-module. Its restriction to ${}^0M^0$ is a multiple of the irreducible representation with highest weight $(s(\rho + \lambda) - \rho)|\mathfrak{b}_c = s(\rho + \lambda)|\mathfrak{b}_c - \rho_{{}^0M}$.

Proof. (4.2) implies that

$$\begin{aligned} I \otimes F_\lambda &= \mathrm{Ind}_P^G(H_\sigma \otimes \mathbb{C}_{\rho_P+\nu}) \otimes F_\lambda \\ &\cong \mathrm{Ind}_P^G(F_\lambda \otimes H_\sigma \otimes \mathbb{C}_{\nu+\rho}). \end{aligned}$$

$H^*(\mathfrak{g}, \mathfrak{k}; V_{(\mathfrak{k})}) \cong H^*(\mathfrak{g}, \mathfrak{k}; V)$ for V a $\mathfrak{g}$-module (as $C^*(\mathfrak{g}, \mathfrak{k}; V_{(\mathfrak{k})}) \cong C^*(\mathfrak{g}, \mathfrak{k}; V)$), so we can replace a differentiable module by its space of K-finite vectors when computing cohomology.

There is a theorem from section 2 stating that for W a differentiable admissible representation of M with W Hilbert, and W_0 the $(\mathfrak{m}, K_P)$-module of K_P-finite vectors in W:

$$H^*(\mathfrak{g}, K; I(W_0)) \cong H^*(\mathfrak{p}, K_P; W_0).$$

Then

$$H^*(\mathfrak{g}, K; I \otimes F_\lambda) = H^*(\mathfrak{p}, K_P; F_\lambda \otimes H_\sigma \otimes \mathbb{C}_{\nu+\rho}).$$

By definition, $\mathfrak{n}$ acts trivially on $H_\sigma \otimes \mathbb{C}_{\nu+\rho}$, so by the Kunneth rule

$$H^*(\mathfrak{n}; F_\lambda \otimes H_\sigma \otimes \mathbb{C}_{\rho+\nu}) = H^*(\mathfrak{n}; F_\lambda) \otimes H_\sigma \otimes \mathbb{C}_{\rho+\nu}.$$

Now we apply (4.7) with $\mathfrak{g} = \mathfrak{p}, L = K_P, K_1 = \{1\}, V = F_\lambda \otimes H_\sigma \otimes \mathbb{C}_{\rho+\nu}$ to get a spectral sequence (E_r) converging to $H^*(\mathfrak{p}, K_P; F_\lambda \otimes H_\sigma \otimes \mathbb{C}_{\rho+\nu})$ and in which

$$\begin{aligned} E_2^{p,q} &= H^p(\mathfrak{p}/\mathfrak{n}, K_P/\{1\}; H^q(\mathfrak{n}, \{1\}; F_\lambda \otimes H_\sigma \otimes \mathbb{C}_{\rho+\nu})) \\ &= H^p(\mathfrak{m}, K_P; H^q(\mathfrak{n}; F_\lambda) \otimes H_\sigma \otimes \mathbb{C}_{\rho+\nu}). \end{aligned}$$

By Konstant's Theorem,

$$H^q(\mathfrak{n}; F_\lambda) = \bigoplus_{s \in W^P, l(s)=q} L_s$$

where $L_s = E_{s(\lambda+\rho)-\rho}$.

This implies that

$$E_2^{p,q} = \bigoplus_{s \in W^P, l(s)=q} H^p(\mathfrak{m}, K_P; L_s \otimes H_\sigma \otimes \mathbb{C}_{\rho+\nu})).$$

Since $M = {}^0M \times A$, the M-module L_s may be viewed as a tensor product of an irreducible ${}^0M$-module by the one-dimensional A-module $\mathbb{C}_{s(\rho+\lambda)-\rho)|_A}$.

Let

$$\begin{aligned} \nu_s &= s(\rho+\lambda)|_A - \rho|_A + \rho_P + \nu \\ &= s(\rho+\lambda)|_A + \nu \end{aligned}$$

since $\rho|_A = \rho_P$ as (P, A) is standard.

Using the Kunneth rule and the fact that $H^q(\mathfrak{g}, K; V) = H^q(\mathfrak{g}, \mathfrak{k}; V)^{K/K^0}$, we get

$$\begin{aligned} H^*(\mathfrak{m}, K_P; L_s \otimes H_\sigma \otimes \mathbb{C}_{\nu_s}) &= H^*(\mathfrak{m}, \mathfrak{k}_P; L_s \otimes H_\sigma \otimes \mathbb{C}_{\nu_s})^{K_P/K_P^0} \\ &= H^*({}^0\mathfrak{m}, K_P; L_s \otimes H_\sigma)^{K_P/K_P^0} \otimes H^*(\mathfrak{a}, \mathbb{C}_{\nu_s}) \\ &= H^*({}^0\mathfrak{m}, K_P; L_s \otimes H_\sigma) \otimes H^*(\mathfrak{a}, \mathbb{C}_{\nu_s}). \end{aligned}$$

It is a fact that if U, V are $(\mathfrak{g}, \mathfrak{k})$-modules with $\chi_U \neq \chi_V$, then $\mathrm{Ext}^q(U, V) = 0$ for all q, and $H^q(\mathfrak{g}, U \otimes V) = 0$ in the case that U is finite dimensional and $\chi_{\tilde{U}} \neq \chi_V$ ($\tilde{U}$ is the contragradiant module, the space of K-finite vectors in the dual U').

This imples that if $\nu_s \neq 0$, then $H^*(\mathfrak{a}; \mathbb{C}_{\nu_s}) = 0$. Therefore $E_2 = 0$ and we have completed the proof of (1).

Now suppose that $\nu_s = 0$. Then

$$H^*(\mathfrak{a}, \mathbb{C}_{\nu_s}) = \Lambda \mathfrak{a}_c^*$$

and we have

$$H^*(\mathfrak{m}, K_P; L_s \otimes H_\sigma \otimes \mathbb{C}_{\nu_s}) = H^*({}^0\mathfrak{m}, K_P; L_s \otimes H_\sigma) \otimes \Lambda \mathfrak{a}_c^*.$$

It is also a fact that for U, V $(\mathfrak{g}, K)$-modules, $\chi_U \neq \chi_V$ implies $\mathrm{Ext}_{\mathfrak{g},K}(U, V) = 0$, and if U is finite dimensional, $\chi_{\tilde{U}} \neq \chi_V$ imples $H^q(\mathfrak{g}, K; U \otimes V) = 0$.

This implies that $H^*({}^0\mathfrak{m}, K_P; L_s \otimes H_\sigma) = 0$ if $\chi_\sigma \neq \chi_{\tilde{L}_s}$. Since the highest weight of L_s is $(s(\rho+\lambda) - \rho)|_{\mathfrak{b}_c}$ and $\rho|_{\mathfrak{b}_c} = \rho_{{}^0M}$, the infinitesimal character of $\tilde{L}_s$ is $\chi_{-(s(\lambda+\rho))|_{\mathfrak{b}_c}}$.

This proves (2).

These conditions determine $s(\rho+\lambda)$ uniquely; but $\rho+\lambda$ is regular, so they fix $s \in W$ also, and the uniqueness assertion in (a) follows as $\mathfrak{h}_c^* = \mathfrak{a}_c^* + \mathfrak{b}_c^*$.

Now let $s \in W^P$ satisfy (1) and (2).

We have

$$H^*(\mathfrak{m}, K_P; L_t \otimes H_\sigma \otimes \mathbb{C}_{\rho+\lambda}) = 0$$

for $t \in W^P$ with $t \neq s$ by uniqueness.

This implies that

$$E_2^{p,q} = 0$$

if $q \neq l(s)$.

Further, we have shown

$$E_2^{p,l(s)} = (H^*({}^0\mathfrak{m}, K_P; L_s \otimes H_\sigma) \otimes \Lambda\mathfrak{a}_c^*)^p$$

which shows that the spectral sequence (E_r) degenerates $(E_\infty = E_2)$ and we have

$$H^j(\mathfrak{g}, K; I \otimes F_\lambda) = E_2^{k-l(s),l(s)}.$$

which implies

$$H^{q+l(s)}(\mathfrak{g}, K; I \otimes F_\lambda) = E_2^{q,l(s)} = (H^*({}^0\mathfrak{m}, K_P; H_\sigma \otimes E_{(s(\rho+\lambda)-\rho)})$$

which proves (3).

□

4.3 Fundamental parabolic subgroups

Let L be a reductive group of connected type ($\mathrm{Ad}\, L \subset \mathrm{Ad}\, \mathfrak{g}_c$). Let L_1 be the greatest normal semisimple group of L^0.

(4.11) Definition. A Cartan subgroup C of L is ***fundamental*** iff it contains a maximal torus in L. The fundamental Cartan subgroups of L form a conjugacy class.

A parabolic subgroup P of L is ***fundamental*** if it is minimal among those containing a fundamental Cartan subgroup. Equivalently, if $P \cap L_1$ is fundamental in L_1.

These parabolic subgroups form a class of associated parabolic subgroups: if C is a fundamental Cartan subgroup of L^0 and C_d^0 is the greatest connected $\mathbb{R}$-split subgroup, then $\mathcal{Z}_{L^0}(C_d^0)$ is a Levi subgroup of P for all fundamental parabolic subgroups of L^0 containing C.

In particular, $\mathrm{prk}\, P$ is equal to $\mathrm{rk}\, L - \mathrm{rk}\, Q$ where Q is a maximal compact subgroup of L. If $\mathrm{rk}\, L = \mathrm{rk}\, Q$ (i.e. L has a discrete series), L is its own fundamental subgroup.

(P, A) is ***cuspidal*** if ${}^0M_P$ has a compact Cartan subgroup. If so, the center of ${}^0M_P$ is compact. A fundamental parabolic subgroup if cuspidal.

(4.12) Proposition. Let $\mathfrak{j}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ with associated $J \leq G$ Cartan subgroup. If $\mathfrak{j}$ is fundamentral, then J is connected and abelian.

Proof. To say that $\mathfrak{j}$ is fundamental is to say that $(\mathfrak{g}_c, \mathfrak{j}_c)$ admits no real root (i.e. $(\overline{\mathfrak{g}_c}, \overline{\mathfrak{j}_c})$ admits no real root) which entails that $\overline{\mathfrak{j}_\mathfrak{k}}$ is maximal abelian in $\overline{\mathfrak{k}}$ and so $\mathfrak{i}_\mathfrak{k} = \mathfrak{i}_\mathfrak{k} + c_\mathfrak{k}$ is maximal abelian in $\mathfrak{k}$ — therefore $J_K = J \cap K$ is a maximal torus in K. Notice $J = J_K J_\mathfrak{p}$ for $J_\mathfrak{p} = \exp(\mathfrak{j} \cap \mathfrak{p}) = \exp(\mathfrak{k}_\mathfrak{p})$. Let $Z(J_\mathfrak{p})$ denote the set of $j \in J_K$ such that $i_c(j) \in \exp(\sqrt{-1}\mathfrak{j}_\mathfrak{p})$ for every complexification G_c of G. It is a fact that $J_K = Z(J_\mathfrak{p})J_K$. As the elements of

$J'_K = J \cap K$ all commute with the elements of J_K, $J_K = J'_K$, it follows that the Cartan subgroups are equal. □

(4.13) Remark. If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{z}(\mathfrak{t})$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$, then $\mathfrak{z}(\mathfrak{t})$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$.

(4.14) Lemma. Let (P, A) be a cuspidal p-pair in G, $M = \mathcal{Z}(A)$, $N = R_u P$.

1. If P is fundamental, then all root spaces in $\mathfrak{n}$ are even dimensional. In particular, $\dim \mathfrak{n}$ is even. Moreover
$$\dim \mathfrak{n} \geq \max(2 \dim A, 2 \operatorname{rk} K).$$
2. If P is not fundamental, then the Cartan subalgebras of ${}^0\mathfrak{m}_c$ are singular in $\mathfrak{g}_c$ (this might mean killing form is degenerate (Cartan's Criterion does not hold)).

Proof. 1. Let P be fundamental and let S be a maximal torus of ${}^0M$. Then S is also a Cartan subgroup in a maximal compact subgroup of G; hence it contains elements which are regular in $\mathfrak{g}_c$ (elements with centralizer of smallest possible dimension) by (4.13), and the Cartan subgroup $S \cdot A$ is the centralizer of some element in S.

In particular the representation of S in $\mathfrak{n}$ given by the adjoint representation does not contain a trivial representation. Therefore it is a sum of two-dimensional real irreducible representations ($\mathbb{C}[G] = \mathbb{R}[G] + i\mathbb{R}[G]$ so every irreducible real representation appears in the restriction of an irreducible complex representation, which is 1-dimensional over $\mathbb{C}$ for G abelian, so 2-dimensional over $\mathbb{R}$).

Since S leaves all root spaces stable, this proves (1) and also shows that $\dim \mathfrak{n} \geq 2 \dim S = 2 \cdot \operatorname{rk} K$. Since A acts faithfully on $\mathfrak{n}$, there are at least $\dim A$ linearly independent roots, hence $\dim \mathfrak{n} \geq 2 \dim A$.

2. Assume P is not fundamental. Let T be a maximal torus of G containing S. Then $T \subset \mathcal{Z}(S)$ and $T \neq S$. The group $R = \mathcal{Z}(S)/S$ is reductive. The group A maps isomorphically onto the identity component of a Cartan subgroup of R and is $\mathbb{R}$-split. But R contains a nontrivial torus T/S and hence its Cartan subgroups are not all conjugate to each other. This means that R is not commutative and so $Z(S)$ has a nontrivial semisimple subgroup. But then $\mathfrak{s}$ is singular. Since $\mathfrak{s}_c$ is a Cartan subgroup of ${}^0\mathfrak{m}_c$, this proves (2).

□

Let L be a Lie group with finitely many connected components and maximal compact subgroup Q. Put
$$2q(L) := \dim L - \dim Q.$$
Assume $\mathfrak{l}$ is reductive. Then let
$$l_0(L) = \operatorname{rk} L - \operatorname{rk} Q$$
$$2q_0(L) = 2q(L) - l_0(L).$$
Since the rank and dimension of a reductive Lie algebra are congruent $\mod 2$, $q_0 \in \mathbb{Z}$.

(4.15) Lemma. Let L be reductive with compact center. Then $q_0(L) \geq \mathrm{rk}_{\mathbb{R}} L$ and $q_0(L) + l_0(L) \leq 2q(L) - \mathrm{rk}_{\mathbb{R}} L$.

Proof. We can assume the L is connected. Let $L = L'S$ with S central compact, L' semisimple. $L' \cap S$ finite.

$q_0, l_0, \mathrm{rk}_{\mathbb{R}}, q$ are the same for L and L' so we reduce to the case that L is connected and semisimple. We write $L = G$ as passing to finite covering leaves constants invariant.

The set ${}_{\mathbb{R}}\Delta$ has $\dim A_0$ elements, hence $\dim N_0 \geq \dim A_0$. By the Iwasawa decomposition $G = KA_0N_0$, we have

$$2q(G) = \dim A_0 + \dim N_0 + \dim K - \dim K = \dim A_0 + \dim N_0 \geq 2\dim A_0 = 2\,\mathrm{rk}_{\mathbb{R}}\, G. \tag{10}$$

This proves the lemma when $l_0(G) = 0$.

Let (P, A) be a standard fundamental p-pair of G, $P = MN$ Levi decomposition, S maximal torus in ${}^0M$. ${}^0M$ has compact center; hence (10) also yields

$$q({}^0M) \geq \mathrm{rk}_{\mathbb{R}}({}^0M). \tag{11}$$

We have

$$q({}^0M) \geq \mathrm{rk}_{\mathbb{R}}({}^0M). \tag{12}$$

We have

$$\begin{aligned} \mathrm{rk}_{\mathbb{R}}\, G &= \mathrm{rk}_{\mathbb{R}}({}^0M) + \dim A, \\ \dim A &= l_0(G). \end{aligned}$$

As P is standard, the Iwasawa decomposition $G = KA_0N_0$ induces one on ${}^0M$, so

$$2q(G) = 2q({}^0M) + \dim N + \dim A = 2q({}^0M) + \dim N + l_0(G), \tag{13}$$

$$2q_0(G) = 2q({}^0M) + \dim N. \tag{14}$$

Using (11), (4.14), and (12), we get

$$q_0(G) \geq \mathrm{rk}_{\mathbb{R}}({}^0M) + (\dim N)/2 \geq \mathrm{rk}_{\mathbb{R}}({}^0M) + \dim A = \mathrm{rk}_{\mathbb{R}}\, G. \tag{15}$$

On the other hand, by (13) and (14), we get

$$2q(G) - \mathrm{rk}_{\mathbb{R}}\, G = 2q({}^0M) + \dim N - \mathrm{rk}_{\mathbb{R}}({}^0M) = 2q_0(G) - \mathrm{rk}_{\mathbb{R}}({}^0M)$$

and then (15) yields

$$2q(G) - \mathrm{rk}_{\mathbb{R}}\, G \geq q_0(G) + \mathrm{rk}_{\mathbb{R}}\, G - \mathrm{rk}_{\mathbb{R}}({}^0M) = q_0(G) + l_0(G).$$

□

(4.16) Lemma. Let L be a reductive group with identity component with compact center.

1. $q(L) = \mathrm{rk}_{\mathbb{R}} L$ iff the non-compact normal subgroups of L^0 are of type $\mathrm{SL}_2(\mathbb{R})$.

2. $q_0(L) = \mathrm{rk}_{\mathbb{R}} L$ iff every non-compact simple factor of L^0 is of type $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SL}_3(\mathbb{R})$.

Proof. We can reduce to the case L is connected, simple, noncompact. Fix a minimal p-pair (P_0, A_0) and let $P_0 = M_0 N_0$ be the standard Levi decomposition of P_0.

1. By (1) in the previous lemma, $q(L) = \mathrm{rk}_{\mathbb{R}} L$ is equivalent to

$$\dim N_0 = \mathrm{rk}_{\mathbb{R}} L = \dim A_0.$$

   Since L is simple, and $\dim N_0 \geq \mathrm{Card}_{\mathbb{R}}\, \Phi^+$, this is possible only if $\dim A_0 = 1$.

   Also $\mathrm{rk}\, L = 1$ as any maximal torus of M_0 acts trivially on the one dimensional space N_0, so is reduced to $\{1\}$. Then L is locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ (have the same Lie algebras). The converse is clear.

2. Let $q(L) \neq q_0(L)$ and $q_0(L) = \mathrm{rk}_{\mathbb{R}} G$. Let (P, A) be a standard fundamental p-pair. P is cuspidal, hence $q({}^0M) = q_0({}^0M)$. By (14), the previous lemma, and (4.14),

$$q_0(L) = q({}^0M) + (\dim N)/2$$
$$q({}^0M) \geq \mathrm{rk}_{\mathbb{R}}({}^0M) \tag{16}$$
$$\dim N \geq 2 \dim A. \tag{17}$$

   By (15)

$$q_0(L) = \mathrm{rk}_{\mathbb{R}} L \quad \Leftrightarrow \quad q({}^0M) = \mathrm{rk}_{\mathbb{R}}({}^0M)$$
$$\dim N = 2 \dim A.$$

   By part (1), (17) is equivalent to ${}^0M$ having all of its noncompact simples factors of type $\mathrm{SL}_2(\mathbb{R})$. In view of (4.14), (16) yields

$$\Phi(P, A) = \Delta(P, A). \tag{18}$$

   Now assume L is absolutely simple (no proper nontrivial series subgroups ($H \leq G$ with a chain of subgroups from H to G such that each consective pair of subgroups is normal)). Then (18) implies by standard facts about roots that $\dim A = 1$, so by (4.14), $\mathrm{rk}({}^0M) \leq 1$. If $\mathrm{rk}({}^0M) = 0$, then L is of type $\mathrm{SL}_2(\mathbb{R})$, and $q(L) = q_0(L)$, in contradiction to our assumption. Hence $\mathrm{rk}({}^0M) = 1$, so $\mathrm{rk}\, L = 2$.

   The representation of ${}^0M^0$ is either a circle group or locally adjoint with finite kernel. In the former case, no root of L restricts to zero on $\mathfrak{a}$ and $\Phi(P, A)$ has at least two elements. Therefore ${}^0M^0$ is of type $\mathrm{SL}_2(\mathbb{R})$.

   We have a semidirect product decomposition $N_0 = N({}^0M \cap N_0)$ where $\dim({}^0M \cap N_0) = 1$, hence $\dim N_0 = 3$. From this it follows that L is locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$.

   Finally, suppose L is not absolutely simple. Then there exists an absolutely simple complex group R such that L is R as a real Lie group. In this case, $\Phi(P, A)$ may be viewed as the set of positive roots in $\Phi(R)$. Then (18) shows that R has rank 1; i.e. R is locally isomorphic to $\mathrm{SL}_2(\mathbb{C})$.

□

4.4 Tempered Representations

(4.17) Theorem. Let (P, A) be a standard cuspidal p-pair of G and (σ, H_0) a discrete series representation of ${}^0M$, and let $\nu \in \mathfrak{a}_c^*$ be purely imaginary. Let $I = I_{P,\sigma,\nu}$ be as before.

Assume that $H^*(\mathfrak{g}, K; I \otimes F_\lambda) \neq 0$. Then

- $\nu = 0$;
- P is fundamental;
- $\dim H^q(\mathfrak{g}, K; I \otimes F_\lambda) = \binom{l_0}{q-q_0}$ for $q \in \mathbb{N}$, $q_0 = q_0(G)$, $l_0 = l_0(G)$;
- $H^q(\mathfrak{g}, K; I \otimes F_\lambda) = 0$ for $q \notin [q_0, q + l_0]$;
- $l(s) = (\dim N)/2$ for s the unique element from before.

Proof. By (4.8), the nonvanishing of cohomology implies that ν is real, so $\nu = 0$ as it is totally imaginary.

Then by (4.8)(1), $s(\rho + \lambda)|_A = 0$, so $s(\rho + \lambda) \in \mathfrak{b}_c^*$. Since $s(\rho + \lambda)$ is regular (a weight γ is regular if $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$), it follows that $\mathfrak{b}_c^*$ is not orthogonal to any root, or equivalently, $\mathfrak{b}$ contains regular elements of $\mathfrak{g}_c$.

Then (4.14)(2) shows that P is fundamental (as otherwise the Cartan subalgebra $\mathfrak{b}$ of ${}^0\mathfrak{m}_c$ would be singular in $\mathfrak{g}_c$ but the killing form must be nondegenerate if each element is regular so this is not the case). Then

$$\dim A = \operatorname{prk} P = l_0(G) = \operatorname{rk} G - \operatorname{rk} K. \tag{19}$$

We will now use (4.8)(3), writing $L_s = E_{s(\rho+\lambda)-\rho}$.

We have assumed that σ belongs to a discrete series of ${}^0M$. $\mathrm{Ext}_{{}^0\mathfrak{m},\mathfrak{k}_P}(L_s^*, H) = H^i({}^0\mathfrak{m}, \mathfrak{k}_P; L_s \otimes H)$ so by (2.14) and by (2.15), $\mathrm{Ext}^i_{{}^0\mathfrak{m}}(L_s^*, H) = 0$ for $i \neq (\dim^0 M/K)/2 = q({}^0M)$ and has dimension 1 at $q({}^0M)$ as it is nonzero.

Then

$$\begin{aligned} H^{q+l(s)}(\mathfrak{g}, K; I \otimes F_\lambda) &= (H^*({}^0\mathfrak{m}, K_P; H \otimes E_{s(\rho+\lambda)-\rho}) \otimes \textstyle\bigwedge \mathfrak{a}_c^*)^q \\ &= \textstyle\bigwedge^j \mathfrak{a}_c^*. \end{aligned}$$

for $j = q - q({}^0M)$.

This implies that

$$\dim H^q(\mathfrak{g}, K; I \otimes F_\lambda) = \binom{l_0}{q - q_0}$$

In particular the highest and lowest dimensions such that the LHS is nonzero are $q(^0M) + l(s)$ and $q(^0M) + l(s) + l_0(G)$.

The representation I is unitary since σ is, and ν is purely imaginary. All irreducible unitary representations are admissible (admissible meaning restriction to K is unitary and each irreducible unitary representation of K occurs in it with finite multiplicity). Therefore $H^*(\mathfrak{g}, K; I \otimes F_\lambda)$ satisfies Poincare dualty ($H^q(\mathfrak{g}, \mathfrak{k}; V) = H^{2q(G)-q}(\mathfrak{g}, \mathfrak{k}; V)$) so $q(^0M) + l(s) = 2q(G) - (q(^0M) + l(s) + l_0(G))$ so

$$2q(^0M) + 2l(S) + l_0(G) = 2q(G)$$

Then (19) and the fact that $2q(G) = 2q(^0M) + \dim N + \dim A$, we have

$$\dim N = 2l(s)$$

so $l(s) = \dim N/2$.

□

(4.18) Example. A simple example is that if we let F be an imaginary quadratic field, $\mathcal{G} = R_{F/\mathbb{Q}} \mathrm{SL}_2$, the restriction of scalars of SL_2 from F to $\mathbb{Q}$, then we have $F \times \mathbb{R} \cong \mathbb{C}$, so $G \cong \mathrm{SL}_2(\mathbb{C})$, $K \cong \mathrm{SU}(2)$, $A_G \cong \{1\}$, and $G/K \cong \mathfrak{H}_3$ for $\mathfrak{h}_3$ three dimensional hyperbolic space. Then

$$\begin{aligned} 2q &= \dim \mathfrak{H}_3 = 3 \\ l_0 &= \operatorname{rk} \mathrm{SL}_2(\mathbb{C}) - \operatorname{rk} \mathrm{SU}(2) = 2 - 1 = 1 \\ 2q_0 &= 2q - l_0 = 3 - 1 = 2 \end{aligned}$$

so

$$\dim H^q(\mathfrak{sl}_2(\mathbb{C}), \mathrm{SU}(2); I \otimes F) = \binom{1}{q-1} = \begin{cases} 1 & \text{if } q = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

(4.19) Remark. It is a fact that linear reductive G has a discrete series representation if and only if $\operatorname{rk} G = \operatorname{rk} K$, that is, $l_0 = 0$.

(4.20) Corollary.

1. $H^q(\mathfrak{g}, K; I \otimes F_\lambda) = 0$ if $q < \operatorname{rk}_{\mathbb{R}} G$ or $q > 2q(G) - \operatorname{rk}_{\mathbb{R}} G$.
2. If $H^q(\mathfrak{g}, K; I \otimes F_\lambda) \neq 0$ for $q = \operatorname{rk}_{\mathbb{R}} G$, then each compact factor of G is isomorphic to $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$, or $\mathrm{SL}_3(\mathbb{R})$.
3. Let (π, V) be an irreducible tempered $(\mathfrak{g}, K)$-module. Then $H^*(\mathfrak{g}, K; V \otimes F_\lambda) = 0$ if $q \notin [q_0(G), q_0(G) + l_0(G)]$.

If V is not a fundamental principal series representation, then $H^*(\mathfrak{g}, K; V \otimes V_\lambda) = 0$.

Proof.

1. This follows immediately from (4.15).

2. This follows from (4.16).

3. If V is irreducible tempered, then it is a direct summand of some $I = I_{P,\sigma,\nu}$ with σ, ν having the desired properties. Then $H^q(\mathfrak{g}, K; V \otimes F_\lambda)$ is a direct summand of $H^*(\mathfrak{g}, K; I \otimes F_\lambda$ and the result follows.

□

5 Venkatesh's conjecture

For this section we used a survey of [5] and also notes on the work of Venkatesh from The London Number Theory Study Group available on Ashwin Iyengar's website.

Now we have that

$$\dim H^{q_0+q}_{\text{temp}}(\Gamma, \mathbb{C}) = \binom{l_0}{q} \dim H^{q_0}_{\text{temp}}$$

and this same inequality continuous even just considering the λ-eigenspaces of a given Hecke operator T. So the spectrum of the Hecke algebra $\mathbb{T}$ acting on the tempered cohomology is degenerate (the same eigenvalues occur in different degrees) so we may look for any extra symmetries to explain this degeneracy, such as $\mathbb{T}$-invariant endomorphisms $H^a_{\text{temp}}(\Gamma, \mathbb{Q}) \to H^b_{\text{temp}}(\Gamma, \mathbb{Q})$.

Reformulated, we want to construct a natural $\mathbb{Q}$-vector space V of dimension l_0 and construct a natural action of $\bigwedge^q V$ on $H^q_{\text{temp}}(\Gamma, \mathbb{Q})$ over which H^q_{temp} is freely generated of dimension q_0.

The reason we can't adapt our proofs from before to construct this is that they use differential forms and work only with complex coefficients. It is hard to produce natural $\mathbb{T}$-endomorphisms of cohomology with $\mathbb{Q}$-coefficients which change the degree. For example, in many cases $H^1(\Gamma, \mathbb{Q}) = 0$ so we wouldn't be able to shift the degree by 1 by using a cup product with an element of it.

(5.1) Conjecture. V in the above is the $\mathbb{Q}$-linear dual of the motivic cohomology group of the $\mathbb{Q}$-motive attached to the adjoint L-function. Moreover, one can explicitly construct the action of $V \otimes \mathbb{Q}_p$ or $V \otimes \mathbb{C}$ on cohomology with $\mathbb{Q}_p$ or $\mathbb{C}$-coefficents.

To suggest how one could produce the extra endomorphisms for complex cohomology, let G be a real reductive algebraic group, $\Gamma \leq G$. Then $\mathfrak{a}_G$ is a l_0-dimensional complex vector space and we can construct, using explicit constructions with differential forms, an action of $\bigwedge^q \mathfrak{a}^*_G$ on $H^q_{\text{temp}}(\Gamma, \mathbb{C})$ (see [6]).

Could put in constructing endomorphisms for $\mathbb{Q}_p$ with derived Hecke operators.

For simplicity assume that $\mathcal{G}$ is simply connected and split. It has an associated Langlands group $\hat{G}$, which we can regard as a split algebraic group over $\mathbb{Q}$. To define this group, we consider the root datum of $\mathcal{G}$. That is, $(X^*, \Phi, X_*, \Phi^\vee)$ where X^* is the lattice of characters of the maximal torus $(T \to \mathbf{G}_m)$, X_* is the dual lattice $(\mathbf{G}_m \to T)$, and Φ is the set of roots apearing in the representation of T on $\mathfrak{g}$, and $\Phi^\vee$ the corresponding coroots. A connected split reductive algebraic group is uniquely determined by its root datum and conversely, for any root datum there is a corresponding connective reductive group. We then define the Langlands dual group to be the connected reductive group with root datum $(X_*, \Phi^\vee, X^*, \Phi)$.

Fix some algebraic closure $\bar{\mathbb{Q}}_l$. It is conjectured in [2] that there is a Galois representation

$$\rho_\pi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \hat{G}(\bar{\mathbb{Q}}_l)$$

categorised by a compatibility between its values at Frobenius elements and the eigenvalues of Hecke operators. We want to construct a linear representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ from this. To do this, compose it with the representation of $\hat{G}$ given by the conjugation action of $\hat{G}$ on its dual Lie algebra $\mathrm{Ad} : \hat{G} \to \mathrm{GL}(\hat{\mathfrak{g}}^*)$. As $\hat{G}$ is semisimple, the Killing form identifies $\hat{g}$ with $\hat{g}^*$. So we get

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\mathrm{Ad}\circ\rho_\pi} \mathrm{GL}_d(\bar{\mathbb{Q}}_l)$$

where $d = \dim \hat{g} = \dim \hat{G}$. Denote it by $\mathrm{Ad}\,\rho_\pi$, the co-adjoint Galois representation attached to π. We know that this ρ_π exists in most cases.

Now in this situation the Langlands program predicts that ρ_π is also motivic and ideally there is a motive M_π of weight 0 and dimension $\dim \mathcal{G}$ such that the action of the Galois group on the étale cohomology of M gives a representation isomorphic to $\mathrm{Ad}\,\rho_\pi$. This motive would have the property that $L(M, s) = L(\mathrm{Ad}, \pi, s)$, the L-function of M coincides with the L-funtion of the adjoint L-function of π. Let $V := H^q_{\mathrm{mot}}(M_\pi, \mathbb{Q}(1))$, which Beilinson's conjecture relates to the the L-function $L(\mathrm{Ad}, \pi, 1)$.

The Langlands conjectures predict that there is a morphism $V \otimes \mathbb{C} \to \mathfrak{a}$ (the Beilinson regulator), and the p-adic regulator gives a map to Galois cohomology (in the sense of Bloch-Kato) $V \to H^1_f(\mathrm{Ad}\,\rho_\pi(1))$. Take the dual of this map and let $\mathfrak{a}^*_{\mathbb{Q}}$ be the elements of $\mathfrak{a}^*$ mapping to V^*. If, as conjectured, the regulator is an isomorphism, then the following holds

(5.2) Conjecture. Under the action of $\bigwedge^q \mathfrak{a}^*$ on $H^q(\Gamma, \mathbb{C})$, the elements of $\bigwedge^q \mathfrak{a}^*_{\mathbb{Q}}$ preserve $H^q(\Gamma, \mathbb{Q})$.

This would mean that the rational structures of motivic and automorphic cohomology 'line up' which is very interesting as the motivic cohomology group and cohomology group $H^q(\Gamma, \mathbb{Q})$ are very different objects and there is otherwise little to suggest why they would detect each other.

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