The application of Emerton's functor of ordinary parts to Hida Theory

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The first section of this project will provide an introduction to classical Hida Theory, proving some of the main results. In the second section, I shall detail a paper [2] of Emerton's in which he reproves these results using tools from representation theory which allows for a more abstract approach. In the third section, I will present Emerton's construction of the functor of ordinary parts which enables us to further generalise the approach from section 2, and I will derive some properties of it.

1 Hida Theory

Haruzo Hida began to develop his theory of ordinary parts in the 1980's. It has since grown and found numerous applications, for instance in proving that ordinary cusp forms occur in p–adic analytic families in his paper [8] in 1986, and notably in Wiles' proof [10] of the Iwasawa main conjecture for totally real fields in 1990.

We are interested whether we can, in some sense, p–adically interpolate the space of modular forms. More precisely, in the question of whether, given any cusp form of some weight and level, we can find a family of modular forms containing it that varies p–adic analytically over weights. Hida gave a partial answer to this question by instead considering the ordinary parts case.

Hida succeeded in constructing a space so called Λ–adic cusp forms $S(N,\Lambda)$ such that each point comes attached to a family of classical forms of levels Np^r for $r \geq 1$ and of varying weights. This space is acted on by the Hecke operator U_p in a natural way, which allows the construction of a projector $e_p = \lim_{n\to\infty} U_p^{n!}$. We call this the ordinary projector, with image the ordinary part of a module.

In this section we will give an introduction to Hida's work and outline his proofs of the following two important theorems which illustrate the efficacy of this approach:

(1.1) Theorem. For $k \geq 2$, we have

$$\operatorname{rank}_{\mathbb{Z}_p} S_k^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\omega^{-k}; \mathbb{Z}_p) = \operatorname{rank}_{\mathbb{Z}_p} S_2^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\omega^{-2}; \mathbb{Z}_p).$$

(1.2) Theorem. The space $S^{\mathrm{ord}}(N,\Lambda)$ is free of finite rank over Λ, and in fact

$$\operatorname{rank}_{\Lambda} S^{\mathrm{ord}}(\chi,\Lambda) = \operatorname{rank}_{\mathbb{Z}_p} S_2^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\omega^{-2}; \mathbb{Z}_p).$$

Finally, there is also the following control theorem which shows that $S_k(\chi,\Lambda)$ is the interpolating space that we wanted:

(1.3) Theorem. For each $k \geq 2$, if $P_k = (X - (u^k - 1)) \subseteq \Lambda$, then evaluation at $u^k - 1$ induces an isomorphism

$$S^{\mathrm{ord}}(\chi,\Lambda)/P_k \xrightarrow{\sim} S_k^{\mathrm{ord}}(\Gamma_0(p), \chi\omega^{-k}; \mathbb{Z}_p).$$

1.1 Λ–adic Forms

(1.4) Notation. Let p be an odd prime and let N be an integer coprime to p.

Let $\Lambda = \mathbb{Z}_p[[X]]$ be the Iwasawa algebra. Let K be a finite field extension of its quotient field, and let I denote the integral closure of Λ in K.

- If $p = 2$, set $q = 4$;
- If $p > 2$, set $q = p$, we will mostly work in the case but everything can be easily generalised to the $p = 2$ case.

Let $W = 1 + q\mathbb{Z}_p$ with topological generator u. There is a decomposition $\mathbb{Z}_p^\times = (\mathbb{Z}/q)^\times \times W$ and, with respect to this, write $x = \omega(x) \times \langle x \rangle$ (so $\omega : \mathbb{Z}_p^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ is a Dirichlet character with conductor q, for p odd it is the mod p cyclotomic character).

Additionally, let χ_ζ denote a Dirchlet character of conductor p^r associated to $\zeta \in \mu_{p^{r-1}}$ with p odd (or $\zeta \in \mu_{p^{r-2}}$ for $p = 2$) defined by mapping the image of $u \in W$ in $(\mathbb{Z}/p^r\mathbb{Z})^\times$ to ζ.

(1.5) Definition (I–adic forms)**.** For p odd[1], the ***I*–*adic form*** F of tame level N and character $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathbb{C}^\times$ is a formal q–expansion

$$F = \sum_{n=0}^{\infty} a_n(F) q^n \in I[[q]]$$

such that for all specialisations $\nu : I \to \overline{\mathbb{Q}}_p$ extending the specialisations given by

$$\begin{aligned} \nu_{k,\zeta} : \Lambda &\to \overline{\mathbb{Q}}_p \\ X &\mapsto \zeta u^k - 1 \end{aligned}$$

for $k > 1$ and $\zeta \in \mu_{p^{r-1}}$ with $r \geq 1$, the specialised q–expansion

$$f_\nu = \nu(F) = \sum_{n=0}^{\infty} \nu(a_n(F)) q^n \in \overline{\mathbb{Q}}_p[[q]]$$

is the image under a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ of the q–expansion in $\overline{\mathbb{Q}}[[q]]$ of a classical modular form of:

- weight k
- level Np^r
- character $\chi_\nu = \chi\omega^{-k}\chi_\zeta$.

We denote by $M(N, \chi, I)$ the ***I*–*module of I*–*adic forms of tame level N and character* χ**, the idea being that each $M(N, \chi, I)$ is a family of classical modular forms of varying weights and levels, all with the same residual q–expansion.

Finally we let

$$M(N, I) := \bigoplus_{\chi} M(N, \chi, I)$$

be the ***I*–*module of all I*–*adic forms***.

We will sometimes also refer to these as **Λ–*adic forms***.

[1] We can give a similar definition for $p = 2$ with some adjustments.

(1.6) Definition (Cuspidal I–adic forms)**.** An I–adic form F is ***cuspidal*** if each specialisation, f_ν, is a cusp form.

We write

$$S(N,I) = \bigoplus_{\chi} S(N,\chi,I)$$

for the *I–**module of** I–**adic cusp forms***.

1.2 The Ordinary Projector

(1.7) Definition (Λ–adic Hecke algebra)**.** We will denote by $h_k(\Gamma,\chi,A)$ the Hecke algebra corresponding to $S_k(\Gamma,\chi,A)$ (the subalgebra of $\mathrm{End}_A(S_k(\Gamma,\chi;A))$ generated by U_n for $n \in \mathbb{N}$), and by $H_k(\Gamma,\chi,A)$ the Hecke algebra corresponding to $M_k(\Gamma,\chi;A)$.

Write $H_k^{\mathrm{ord}}(\Gamma,\chi;A) = eH_k(\Gamma,\chi;A)$ and $h_k^{\mathrm{ord}}(\Gamma,\chi;A) = eh_k(\Gamma,\chi;A)$.

$M(N,I)$ and $S(N,I)$ have actions of the Hecke operators U_n for $n \in \mathbb{N}$ given by the usual formulas on the coefficients[2] and this Hecke action commutes with specialisation. The ***Λ–adic Hecke algebra*** $H(N,\Lambda)$ is the Λ–subalgebra of $\mathrm{End}_\Lambda(M(N,\Lambda))$ generated by these Hecke operators.

We say that Λ–adic form F is an ***eigenform*** if it is an eigenfunction for the Hecke operators. Equivalently, if each specialisation f_ν is an eigenform for the Hecke operators.

For U_p–eigenvectors, the original notion of being ordinary is that a modular form f is ordinary if $a_p(f)$ is a p–adic unit, and a Λ–adic form F is ordinary if each f_v is ordinary. For example, the Eisenstein family $\mathcal{E}_\chi$ is ordinary since $A_{p,\chi}(X) = 1$.

But under this definition, we have the problem that the sum of ordinary forms is not necessarily ordinary. Hida rectified this by constructing the ordinary projector.

(1.8) Fact. Let $K/\mathbb{Q}_p$ be finite. Then for any commutative $\mathcal{O}_K$–algebra A of finite rank, and for any $x \in A$, the limit $\lim_{n\to\infty} x^{n!}$ with respect to the p–adic topology on A exists and gives an idempotent of A.

(1.9) Definition (The ordinary projector)**.** Let $K/\mathbb{Q}_p(\chi)$ be finite. We define the ***ordinary projector*** e of the Hecke algebra by

$$e = e_p = \lim_{n\to\infty} U_p^{n!}.$$

(1.10) Remark. If f is an eigenform of U_p with eigenvalue a,

$$ef = \begin{cases} f & \text{if } |a|_p = 1 \\ 0 & \text{if } |a|_p < 1. \end{cases}$$

(1.11) Definition (Ordinary Λ–adic forms)**.** We define the spaces of ***ordinary Λ–adic forms*** as the images of this idempotent:

$$M^{\mathrm{ord}}(N,I) = eM(N,I)$$
$$S^{\mathrm{ord}}(N,I) = eS(N,I).$$

[2]For $F \in M(N,\Lambda)$ the coefficients of the formal q–expansion for T_nF are given by $a_m(T_nF)(X) = \sum_{b|(m,n)} \kappa(\langle b\rangle)(X)\chi(b)b^{-1}a_{mn/b^2}(F)(X)$ where $\kappa(u^s) = (1+X)^s$. Using $\kappa(\langle n\rangle)(u^k-1) = \omega(n)^{-k}n^k$, we can show $a_n(T_nF)(u^k-1) = a_m(T_nF(u^k-1))$.

1.3 Hida's Proof of Weight Independence

The goal of this section is to prove (1.19).

(1.12) Theorem (Eichler-Shimura Isomorphism)**.** Let $k \geq 2$ and $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be of finite index. For fixed $z_0 \in \mathfrak{h}$, we have an isomorphism[3]

$$\begin{aligned} M_k(\Gamma) \oplus \overline{S_k(\Gamma)} &\xrightarrow{\sim} H^1(\Gamma, L(k-2; \mathbb{C})) \\ (f_1, \overline{f_2}) &\mapsto \left(\gamma \mapsto \int_{z_0}^{\gamma z_0} f_1(z)(Xz+Y)^{k-2} dz + \int_{z_0}^{\gamma z_0} \overline{f_2(z)}(Xz+Y)^{k-2} dz \right). \end{aligned}$$

(1.13) Definition (Parabolic cohomology)**.** We can define the ***parabolic cohomology*** (or cusp cohomology) to be the kernel of the restriction map

$$0 \to H^1_P(\Gamma, L(k;\mathbb{C})) \to H^1(\Gamma, L(k;\mathbb{C})) \xrightarrow{\text{res}} \prod_{c \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} H^1(\mathrm{Stab}_\Gamma(c), L(k;\mathbb{C})).$$

However, the kernel of the composition of the Eichler-Shimura isomorphism with the restriction map

$$M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\sim} H^1(\Gamma, L(k-2;\mathbb{C})) \xrightarrow{\text{res}} \prod_{c \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q}))} H^1(\mathrm{Stab}_\Gamma(c), L(k-2;\mathbb{C}))$$

is $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ so, if Γ has finite index, then $H^1_P(\Gamma, L(k-2;\mathbb{C})) \cong S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ as Hecke modules.

(1.14) Theorem. Let $(N, p) = 1$. Then $\mathrm{rank}_{\mathbb{Z}_p}(h_k^{\mathrm{ord}}(\Gamma_1(Np^r); \mathbb{Z}_p))$ is bounded independently of k if $k \geq 2$, $r \geq 1$.

Proof. Write $\Gamma = \Gamma_1(Np^r)$, $n = k-2$.

Let L' be the image of $H^1(\Gamma, L(n;\mathbb{Z})) \hookrightarrow H^1(\Gamma, L(n;\mathbb{R}))$ and let $L = L' \cap H^1_P(\Gamma, L(n;\mathbb{R}))$, a lattice in $H^1_P(\Gamma, L(n;\mathbb{R}))$.

$\mathrm{End}_{\mathbb{Z}} L$ is free of finite rank over $\mathbb{Z}$, and $h_k(\Gamma;\mathbb{Z})$ is by definition a subalgebra of $\mathrm{End}_{\mathbb{Z}} L$. Let $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$, so $h_k(\Gamma;\mathbb{Z}_p) = h_k(\Gamma;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a subalgebra of $\mathrm{End}_{\mathbb{Z}_p}(L_p)$. Thus $h_k^{\mathrm{ord}}(\Gamma;\mathbb{Z}_p)$ is a subalgebra of $\mathrm{End}_{\mathbb{Z}_p}(eL_p)$ and therefore it is enough to prove that the rank of eL_p is bounded independently of n.

We have an exact sequence of Γ–modules

$$0 \to L(n;\mathbb{Z}) \xrightarrow{p} L(n;\mathbb{Z}) \to L(n;\mathbb{Z}/p\mathbb{Z}) \to 0$$

which yields an exact sequence

$$H^1(\Gamma, L(n;\mathbb{Z})) \xrightarrow{p} H^1(\Gamma, L(n;\mathbb{Z})) \to H^1(\Gamma, L(n;\mathbb{Z}/p\mathbb{Z}))$$

so there is an an embedding $H^1(\Gamma, L(n;\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \hookrightarrow H^1(\Gamma, L(n;\mathbb{Z}/p\mathbb{Z}))$.

Note that $L_p/pL_p = L/pL \hookrightarrow L'/pL'$, and L'/pL' is a surjective image of

$$H^1(\Gamma, L(n;\mathbb{Z}))/pH^1(\Gamma, L(n;\mathbb{Z})) = H^1(\Gamma, L(n;\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z},$$

[3] $L(n;A) = \mathrm{Sym}^n(A^2)$ denotes the space of homogeneous degree n polynomials in two variables with coefficients in A. Γ acts on $L(n;A)$ by $\gamma P(X,Y) = P((\det(\gamma)\gamma^{-1}\binom{X}{Y})^t)$.

so it is sufficient to show that the dimension of $eH^1(\Gamma, L(n; \mathbb{Z}/p\mathbb{Z}))$ is bounded independently of n.

We shall construct an embedding $eH^1(\Gamma, L(n; \mathbb{Z}/p\mathbb{Z})) \hookrightarrow eH^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$.

$$i : L(n; \mathbb{Z}/p\mathbb{Z}) \xrightarrow{P(X,Y) \mapsto P(1,0)} \mathbb{Z}/p\mathbb{Z}$$

is a homomorphism of Γ–modules[a], and composing with 1–cocycles gives a morphism of cohomology groups

$$I = i_* : H^1(\Gamma, L(n; \mathbb{Z}/p\mathbb{Z})) \to H^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$$

which we will show induces an isomorphism $eH^1(\Gamma, L(n; \mathbb{Z}/p\mathbb{Z})) \cong eH^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$.

The short exact sequence of Γ–modules

$$0 \to \ker(i) \to L(n; \mathbb{Z}/p\mathbb{Z}) \to \mathbb{Z}/p\mathbb{Z} \to 0$$

gives rise to the long exact sequence

$$H^1(\Gamma, \ker(i)) \to H^1(\Gamma, L(n; \mathbb{Z}/p\mathbb{Z})) \xrightarrow{I} H^1(\Gamma, \mathbb{Z}/p\mathbb{Z}) \to H^2(\Gamma, \ker(i)) \to \cdots \quad (1)$$

If $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, then $\det(\alpha)\alpha^{-1}$ leaves $\ker(i)$ stable[b], and hence U_p acts naturally on $H^q(\Gamma, \ker(i))$. Further, the action of $\det(\alpha)\alpha^{-1}$ on $\ker(i)$ is nilpotent and, since $\Gamma\alpha\Gamma = \amalg_{m=1}^{p-1} \Gamma\alpha \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, the action of U_p on $H^q(\Gamma, \ker(i))$ is nilpotent for $q \geq 1$.

So applying e to (1) gives that $eH^1(\Gamma, \ker(i)) = eH^2(\Gamma, \ker(i)) = 0$ and we have that $eH^1(\Gamma, L(n; \mathbb{Z}/p\mathbb{Z})) \cong eH^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$.

So the rank is bounded by the rank of $eH^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$ which is independent of n. □

[a] For $P(X,Y) = \sum_{i=0}^n a_i X^{n-i} Y^i \in L(n; A)$, $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} P(X,Y) = \sum_{i=0}^n a_i (X - mY)^{n-i} Y^i$ and hence $\left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} P \right)(1,0) = P(1,0)$. But $\gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod p$ for any $\gamma \in \Gamma$, so i is a homomorphism.

[b] If $P(X,Y) = \sum_{i=1}^n X^{n-i} Y^i \in \ker(i)$, then $\det(\alpha)\alpha^{-1} P(X,Y) = \sum_{i=1}^n p^i X^{n-i} Y^i = p(\sum_{i=1}^n p^{i-1} X^{n-i} Y^i) \in \ker(i)$.

(1.15) Remark. Restriction of U_p gives a surjective homomorphism $H_k(\Gamma_1(N); A) \to H_k(\Gamma_0(N); A)$, so the boundedness of the rank of $H_k(\Gamma_0(N); \mathbb{Z}_p)$ follows from that of $H_k(\Gamma_1(N); \mathbb{Z}_p)$.

(1.16) Corollary. $M_k^{\mathrm{ord}}(\Gamma_1(Np); \mathbb{Z}_p)$ has rank bounded independent of k.

Proof. There is a perfect pairing

$$M_k^{\mathrm{ord}}(\Gamma_1(Np), \chi; \mathbb{Z}_p) \times H_k^{\mathrm{ord}}(\Gamma_1(Np), \chi; \mathbb{Z}_p) \to \mathbb{Z}_p$$
$$(f, T) \mapsto a_1(Tf)$$

(similarly for cusp forms) which induces an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(H_k^{\mathrm{ord}}(\Gamma_1(Np), \chi; \mathbb{Z}_p) \cong M_k^{\mathrm{ord}}(\Gamma_1(Np), \chi; \mathbb{Z}_p)$$

By the Eichler-Shimura isomorphism, it suffices to show that $eH^1(\Gamma_1(Np), L(k-2, \mathbb{Z}_p))$ has rank bounded independent of k. The proof of (1.14) shows that the rank of this is bounded by the dimension of $eH^1(\Gamma_1(Np); \mathbb{Z}/p\mathbb{Z}) = eH^1(\Gamma_1(Np), L(2-2, \mathbb{Z}/p\mathbb{Z}))$, so bounded independently of k for $k \geq 2$. $\square$

(1.17) Proposition. $M^{\mathrm{ord}}(N, \Lambda)$ is finitely generated as an Λ–module.

Proof. Under specialisation at $\nu = \nu_{k,1}$, $F \in M^{\mathrm{ord}}(N, \Lambda)$ become elements of $M_k^{\mathrm{ord}}(\Gamma_1(Np), \mathbb{Z}_p)$ which has rank bounded independently of k by (1.16).

We will first prove that M^{ord} is finitely generated as a Λ–module.

Note that $M^{\mathrm{ord}}(N, \Lambda)$ is by definition a submodule of $\Lambda[[q]]$ and so is Λ–torsion-free.

Let W be a finitely generated free submodule of $M^{\mathrm{ord}}(N, \Lambda)$ with basis $\{F_1, \ldots, F_d\}$. Since the F_i are linearly independent over Λ, we can choose $n_i \in \mathbb{Z}$ such that $D(X) = \det(a_{n_i}(F_j)) \neq 0$ in Λ. Now by the Weierstrass preparation theorem[a], we can write this as $D(X) = p^m P(X)U(X)$ and then, as we can only have finitely many zeros in $p\mathbb{Z}_p$, there must be some $k > 1$ such that the specialisation $D(u^k - 1) \neq 0$.

Let $f_i = F_i(u^k - 1)$. By assumption, these modular forms span a free submodule of rank d of $M_k^{\mathrm{ord}}(\Gamma_1(Np), \mathbb{Z}_p)$ which has rank bounded independently of k, and hence W has bounded rank.

Now suppose that $F_1, \ldots, F_r$ is a maximal such set. Then these form a basis for the vector space $M^{\mathrm{ord}}(N, \Lambda) \otimes_\Lambda Q(\Lambda)$ where $Q(\Lambda)$ is the quotient field of Λ. So if $F \in M^{\mathrm{ord}}(N, \Lambda)$, then $F = \sum_{i=1}^r x_i F_i$ for some $x_i \in Q(\Lambda)$ which are the solution to the linear equation $AX = B \in \Lambda^r$ for $A = (a_{n_i}(F_j))$, $X = (x_1, \ldots, x_r)^t$, $B = (a_{n_1}(F), \ldots, a_{n_r}(F))^t$.

Choose n_i such that $D_A = \det(A) \in \Lambda$ is non-zero. Multiplying by the adjoint matrix yields $D_A x = \mathrm{adj}(A)Ax = \mathrm{adj}(A)B \in \Lambda^r$, $D_A x_i \in \Lambda$ for all i, so $D_A M^{\mathrm{ord}}(N, \Lambda) \subseteq \Lambda F_1 + \cdots + \Lambda F_r$ which implies that $M^{\mathrm{ord}}(N, \Lambda) \cong D_A M^{\mathrm{ord}}(N, \Lambda)$ is finitely generated as an Λ–module (since it is isomorphic to a submodule of a finitely generated Λ–module and since Λ is Noetherian). $\square$

[a] (Weierstrass preparation theorem). Let $f(X) = \sum_{i \geq 0} a_i X^i \in \Lambda$ be a power series and assume that $|a_i| < 1$ for $1 \leq i \leq n-1$ and $a_n \in \mathbb{Z}_p^\times$. Then we can uniquely write f in the form $f(X) = P(X)U(X)$ where $U(X)$ is a unit and $P(X)$ is a distinguished polynomial of degree n (i.e. it has the form $P(X) = b_0 + b_1 X + \cdots + X^n$ where $|b_i| < 1$ for all $0 \leq i \leq n-1$).

(1.18) Theorem. $M^{\mathrm{ord}}(N, \Lambda)$ is a finitely generated free Λ–module[4].

Proof. We note first that Λ is a UFD and is a compact ring.

Let $F \in M^{\mathrm{ord}}(N, \Lambda)$. If $F(u^k - 1) = 0$ then $a_n(F)(u^k - 1)$ is divisible by $P = X - (u^k - 1)$ for all k. $(F/P) = F(u^j - 1)/(u^j - u^k)$ for all $j \neq k$ and it is a modular form, so $F/P \in M^{\mathrm{ord}}(N, \Lambda)$. Therefore $PM^{\mathrm{ord}} = \{F \in M^{\mathrm{ord}} \mid F(u^k - 1) = 0\}$ so we can embed $M^{\mathrm{ord}}/PM^{\mathrm{ord}}$ into $M_k^{\mathrm{ord}}(\Gamma_1(Np), \chi\omega^{-k}; \mathbb{Z}_p)$ and $M^{\mathrm{ord}}/PM^{\mathrm{ord}}$ is $\mathbb{Z}_p$–free of finite rank.

Take a $\mathbb{Z}_p$–basis $F_i \mod PM^{\mathrm{ord}}$ of $M^{\mathrm{ord}}/PM^{\mathrm{ord}}$. If $\lambda_1 F_1 + \cdots + \lambda_r F_r = 0$ then we can assume $P \nmid \lambda_i$ for some i and reducing modulo P gives a nontrivial linear relation between the $F_i \mod P$ which is a contradiction, so the F_i's are linearly independent over Λ.

$W = \Lambda F_1 + \cdots + \lambda F_r$ is a free Λ–module of rank r. For any $F \in M^{\mathrm{ord}}$ there is a

[4] This also holds for $S^{\mathrm{ord}}(N, \Lambda)$ with the same proof.

nontrivial linear combination L_0 of the F_j's such that $P|(F - L_0)$. We can repeat this argument with $(F - L_0)/P$ to get another linear combination L_1 with $P|((F - L_0)/P - L_1)$ and so on to get $F = L_0 + L_1 P + \cdots + L_{i-1} P^{i-1} \mod P^i$ which implies that $W/P^i W = M^{\mathrm{ord}}/P^i M^{\mathrm{ord}}$ for all i.

As $W \cong \Lambda^r$, the sequence $L_0 + L_1 P + \cdots + L_{i-1} P^{i-1}$ converges and hence $W = M^{\mathrm{ord}}$ and so M^{ord} is free. □

Now (1.19) follows as a corollary.

(1.19) Theorem. For $k \geq 2$

$$\begin{aligned}
\operatorname{rank}_{\mathbb{Z}_p} H_k^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\omega^{-k}; \mathbb{Z}_p) &= \operatorname{rank}_{\mathbb{Z}_p} M_k^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\omega^{-k}; \mathbb{Z}_p) \\
&= \operatorname{rank}_{\mathbb{Z}_p} M_2^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\omega^{-2}; \mathbb{Z}_p) \\
\operatorname{rank}_{\mathbb{Z}_p} h_k^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\omega^{-k}; \mathbb{Z}_p) &= \operatorname{rank}_{\mathbb{Z}_p} S_k^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\omega^{-k}; \mathbb{Z}_p) \\
&= \operatorname{rank}_{\mathbb{Z}_p} S_2^{\mathrm{ord}}(\Gamma_0(Np^r), \chi\omega^{-2}; \mathbb{Z}_p).
\end{aligned}$$

2 Emerton's Proof of Weight Independence

In 1999, Matthew Emerton demonstrated a new representation theoretic approach to proving certain results of Hida's [2].

He considered the tower of modular curves studied by Hida

$$\cdots \to Y_1(Np^r) \to \cdots \to Y_1(Np) \tag{2}$$

corresponding to the chain of congruence subgroups

$$\cdots \subset \Gamma_1(Np^r) \subset \cdots \subset \Gamma_1(Np)$$

and proved, using only the basic algebraic topology of the $Y_1(p^r)$, the two important theorems of Hida (2.14) and (2.22).

2.1 Congruence Subgroups

Let $p \geq 5$ (everything holds for $p = 3$ if $N > 1$, and can be adapted for $p = 2$) and let $(N, p) = 1$ be such that $\Gamma_1(Np)$ is torsion free.

Taking the homology with $\mathbb{Z}$–coefficients of the chain of modular curves (2) and noticing that $H_1(G, \mathbb{Z}) = G^{\mathrm{ab}}$, we get a chain

$$\cdots \to \Gamma_1(Np^r)^{\mathrm{ab}} \to \cdots \to \Gamma_1(Np)^{\mathrm{ab}}. \tag{3}$$

(2.1) Definition. Let Γ_r be the subgroup of index p^{r-1} in $\Gamma := \Gamma_0$ defined by

$$1 \to \Gamma_r \to \mathbb{Z}_p^\times \to (\mathbb{Z}_p/p^r)^\times \to 1.$$

(2.2) Definition (Intermediate congruence subgroups)**.** For $r \geq s > 0$, define congruence subgroups[5]

$$\Phi_r^s = \Gamma_1(Np^s) \cap \Gamma_0(p^r) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \mod Np^s, c \equiv 0 \mod Np^r \right\}$$

[5] $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \mod N \right\}$ and $\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a - 1 \equiv d - 1 \equiv c \equiv 0 \mod N \right\}$.

with

$$\Gamma_1(Np^r) \subset \Phi_r^s \subset \Phi_r^1 \subset \Gamma_1(Np).$$

In particular, Φ_r^1 is the normaliser of $\Gamma(Np^r)$ in $\Gamma_1(Np)$.

(2.3) Remark. There is a map[6]

$$\begin{aligned} \Phi_r^s &\to \Gamma_s/\Gamma_r \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto d \mod p^r \end{aligned}$$

so Φ_r^s satisfies

$$1 \to \Gamma_1(Np^r) \to \Phi_r^s \to \Gamma_s/\Gamma_r \to 1,$$

so $\Phi_r^s/\Gamma_1(Np^r) = \Gamma_s/\Gamma_r$ and abelianisation gives us

$$\Gamma_1(Np^r)^{\mathrm{ab}} \to \Phi_r^{s\,\mathrm{ab}} \to \Gamma_s/\Gamma_r \to 1.$$

(2.4) Definition (The nebentypus action)**.** The action of Φ_r^1 on $\Gamma_1(Np^r)$ by conjugation induces an action of the quotient $\Phi_r^1/\Gamma_1(Np^r) \cong \Gamma/\Gamma_r$ on $\Gamma_1(Np^r)^{\mathrm{ab}}$. Thus Γ acts on $\Gamma_1(Np^r)^{\mathrm{ab}}$ through Γ/Γ_r and the morphisms in the chain (3) are Γ–equivariant. We call the morphisms induced by the elements of Γ the ***diamond operators***.

(2.5) Remark. The sequence

$$1 \to \Gamma_1(Np^r)/[\Phi_r^s, \Gamma_1(Np^r)] \to \Phi_r^s/[\Phi_r^s, \Gamma_1(Np^r)] \to \Gamma_s/\Gamma_r \to 1$$

is a central extension of the cyclic group Γ_s/Γ_r. It follows that $\Phi_r^s/[\Phi_r^s, \Gamma_1(Np^r)]$ is abelian and so $[\Phi_r^s, \Gamma_1(Np^r)] = [\Phi_r^s, \Phi_r^s]$.

Let $\mathfrak{a}_s$ denote the augmentation ideal of $\mathbb{Z}[\Gamma_s]$. By (2.4)

$$\mathfrak{a}_s \Gamma_1(Np^r)^{\mathrm{ab}} = [\Phi_r^s, \Gamma_1(Np^r)]/[\Gamma_1(Np^r), \Gamma_1(Np^r)]$$

so we can rewrite the extension as

$$1 \to \Gamma_1(Np^r)^{\mathrm{ab}}/\mathfrak{a}_s \to \Phi_r^{s\,\mathrm{ab}} \to \Gamma_s/\Gamma_r \to 1. \tag{4}$$

This implies that a map $\Gamma_1(Np^r)^{\mathrm{ab}} \to \Gamma(Np^s)^{\mathrm{ab}}$ in (3) can be factored as the composition

$$\begin{tikzcd} \Gamma_1(Np^r)^{\mathrm{ab}} \arrow[r, two heads] \arrow[rrr, bend left] & \Gamma_1(Np^r)^{\mathrm{ab}}/\mathfrak{a}_s \arrow[r] & \Phi_r^{s\,\mathrm{ab}} \arrow[r] & \Gamma_1(Np^s)^{\mathrm{ab}}. \end{tikzcd}$$

2.2 Hecke Operators

(2.6) Definition (Hecke operators)**.** Suppose that T is a group with $G, H \leq T$ and suppose that $t \in T$ is such that $t^{-1}Ht \cap G$ has finite index in G. There is a transfer morphism[7] $V : G^{\mathrm{ab}} \to (t^{-1}Ht \cap G)^{\mathrm{ab}}$.

[6]This is surjective as given $\bar{d}$ in Γ_s/Γ_r, we can take a lift $d = 1 + kp^sN$ for some $k \in \mathbb{Z}$ (as if $d_1, d_2 \in \Gamma_s$, then $d_1 \equiv d_2 \mod \Gamma_r$ iff $d_1 - d_2 \in p^r\mathbb{Z}_p$). Then take $c = Np^r$, so $(c, d) = 1$ so there exist $a, b \in \mathbb{Z}$ with $ad - bc = 1$ and $a \equiv d \mod \Gamma_r$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Phi_r^s$.

[7]If G has a finite index subgroup H we can define $V : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$: Take coset representatives $x_1, \dots, x_n$ of H in G. Then, if $y \in G = \bigcup x_j H$, $yx_i = x_j a_i$ for some j and some $a_j \in H$. Define $V(y) = \prod_{i=1}^n a_i \in H^{\mathrm{ab}}$.

Further, conjugation by t induces $(t^{-1}Ht \cap G)^{\mathrm{ab}} \cong (H \cap tGt^{-1})^{\mathrm{ab}}$ and inclusion induces $(H \cap tGt^{-1})^{\mathrm{ab}} \to H^{\mathrm{ab}}$. We can compose these to obtain a morphism

$$\begin{tikzcd} G^{\mathrm{ab}} \arrow[r, "V"] \arrow[rrr, bend right, "{[t]}"'] & (t^{-1}Ht \cap G)^{\mathrm{ab}} \arrow[r, "t(-)t^{-1}"] & (H \cap tGt^{-1})^{\mathrm{ab}} \arrow[r] & H^{\mathrm{ab}}. \end{tikzcd}$$

(2.7) Definition (Atkin U–operator)**.** Take $T = \mathrm{GL}_2(\mathbb{Q})$ and $G = H = \Gamma(Np)$ some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then if $t = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ we denote the corresponding Hecke operator by $U = [t]$, the ***Atkin*** U***–operator***.

(2.8) Remark. Suppose $G = \Phi_r^s$ in (2.7). Then[8]

$$\begin{aligned} t^{-1}\Phi_r^s t \cap \Phi_r^s &= \Phi_r^s \cap \Gamma^0(p) \\ \Phi_r^s \cap t\Phi_r^s t^{-1} &= \Phi_{r+1}^s \end{aligned}$$

so U is defined by the composition

$$\begin{tikzcd} U : \Phi_r^{s\,\mathrm{ab}} \arrow[r, "V"'] \arrow[rr, bend left, "U'"] & (\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \arrow[r, "t(-)t^{-1}"'] & \Phi_{r+1}^{s\,\mathrm{ab}} \arrow[r] & \Phi_r^{s\,\mathrm{ab}}. \end{tikzcd}$$

$\Phi_r^{s\,\mathrm{ab}}$ can be made into a $\mathbb{Z}[U]$–module and the maps $\Phi_r^{s\,\mathrm{ab}} \to \Phi_{r'}^{s'\,\mathrm{ab}}$ induced by inclusions are morphisms of $\mathbb{Z}[U]$–modules. This is because, for $r \geq s > 0$, $r' \geq s' > 0$, $r \geq r'$, $s \geq s'$ the following diagram commutes[9]:

$$\begin{tikzcd} \Phi_r^{s\,\mathrm{ab}} \arrow[r] \arrow[d, "V"] & \Phi_{r'}^{s'\,\mathrm{ab}} \arrow[d, "V"] \\ (\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \arrow[r] \arrow[d, "t(-)t^{-1}"] & (\Phi_{r'}^{s'} \cap \Gamma^0(p))^{\mathrm{ab}} \arrow[d, "t(-)t^{-1}"] \\ \Phi_{r+1}^{s\,\mathrm{ab}} \arrow[r] & \Phi_{r'+1}^{s'\,\mathrm{ab}} \end{tikzcd}$$

Further, the action of U on $\Phi_r^{s\,\mathrm{ab}}$ commutes with the action of Γ on $\Phi_r^{s\,\mathrm{ab}}$ via the diamond operators defined in (2.4). It suffices to show[10] that U commutes with conjugation by any

[8] $\Gamma^0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b \equiv 0 \mod p \right\}$.

[9] The lower half clearly commutes. To show that the upper half commutes, notice that $\Phi_r^s \cap \Gamma^0(p)$ has coset representatives $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ in Φ_r^s for $0 \leq i \leq p-1$ which are independent of r and s, so $\Phi_r^{s\,\mathrm{ab}} \xrightarrow{V} (\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}}$ is given by a formula independent of them which proves commutativity.

[10] Given any $d \in \Gamma/\Gamma_s$, we may find $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \equiv 0 \mod p^{r+1}$ and $b \equiv 0 \mod p$. Then $\langle d \rangle$ is given by conjugation by α.

$\alpha \in \Phi^s_{r+1} \cap \Gamma^0(p)$. The following diagram clearly commutes

$$\begin{tikzcd}
\Phi_r^{s\,\mathrm{ab}} \arrow[rr, "\alpha(-)\alpha^{-1}"] \arrow[d, "V"'] & & \Phi_r^{s\,\mathrm{ab}} \arrow[d, "V"] \\
(\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \arrow[r, "\alpha(-)\alpha^{-1}"] \arrow[d, "t(-)t^{-1}"'] & (\alpha(\Phi_r^s \cap \Gamma^0(p))\alpha^{-1})^{\mathrm{ab}} = & (\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \arrow[d, "\alpha t\alpha^{-1}(-)\alpha t^{-1}\alpha^{-1}"] \\
\Phi_{r+1}^{s\,\mathrm{ab}} \arrow[rr, "\alpha(-)\alpha^{-1}"] \arrow[d] & & (\alpha\Phi_{r+1}^s\alpha^{-1})^{\mathrm{ab}} = \Phi_{r+1}^{s\,\mathrm{ab}} \arrow[d] \\
\Phi_r^{s\,\mathrm{ab}} \arrow[rr, "\alpha(-)\alpha^{-1}"] & & (\alpha\Phi_r^s\alpha^{-1})^{\mathrm{ab}} = \Phi_r^{s\,\mathrm{ab}}
\end{tikzcd}$$

and if $g \in \Phi^s_r \cap \Gamma^0(p)$, then $\alpha t\alpha^{-1}g\alpha t^{-1}\alpha^{-1} = (\alpha t\alpha^{-1}t^{-1})tgt^{-1}(\alpha t\alpha^{-1}t^{-1})^{-1}$ and $\alpha t\alpha^{-1}t^{-1} \in \Gamma(Np^{r+1})$ so conjugation by this is the identity on $\Phi^{s\,\mathrm{ab}}_{r+1}$. Therefore the vertical map on the right is also U and we are done.

(2.9) Lemma. $\Gamma_1(Np^r)^{\mathrm{ab}} \to \Phi^{s\,\mathrm{ab}}_r$ is a morphism of $\mathbb{Z}[U]$–modules and so its cokernel Γ_s/Γ_r is also a $\mathbb{Z}[U]$–module. Further, U acts on Γ_s/Γ_r as multiplication by p.

Proof. Consider the U action of $\Phi^{s\,\mathrm{ab}}_r/\Gamma_1(Np^r)^{\mathrm{ab}}$. We can decompose U as

$$\frac{\Phi_r^{s\,\mathrm{ab}}}{\Gamma_1(Np^r)^{\mathrm{ab}}} \xrightarrow{V} \frac{(\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}}}{(\Gamma_1(Np^r) \cap \Gamma^0(p))^{\mathrm{ab}}} \xrightarrow{t(-)t^{-1}} \frac{\Phi_{r+1}^{s\,\mathrm{ab}}}{\Phi_{r+1}^{r\,\mathrm{ab}}} \to \frac{\Phi_r^{s\,\mathrm{ab}}}{\Gamma_1(Np^r)^{\mathrm{ab}}}$$

where we use coset representatives $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ to define the map V, which turns out to be $\bar{\alpha} \mapsto \bar{\alpha}^p$.

Notice that for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $t\alpha t^{-1} = \begin{pmatrix} a & bp \\ c/p & d \end{pmatrix}$, $t^{-1}\alpha t = \begin{pmatrix} a & b/p \\ cp & d \end{pmatrix}$ all have the same $(1,1)$ and $(2,2)$ entries, so represent the same coset mod $\Gamma_1(Np^r)^{\mathrm{ab}}$.

Thus U acts on $\Phi^{s\,\mathrm{ab}}_r/\Gamma_1(Np^r)^{\mathrm{ab}}$ as $\bar{\alpha} \mapsto \bar{\alpha}^p$, so acts on Γ_s/Γ_r by multiplication by p. $\square$

2.3 Ordinary Parts

(2.10) Definition (Ordinary parts of modules)**.** Take Z an indeterminate with $\mathbb{Z}_p[Z]$ commutative. Let U be an $\mathbb{Z}_p[Z]$–module, finitely generated as an $\mathbb{Z}_p$–module.

If B is the image of the morphism of $\mathbb{Z}_p$–modules $\mathbb{Z}_p[Z] \to \mathrm{End}_{\mathbb{Z}_p}(U)$, then as $\mathrm{End}_{\mathbb{Z}_p}(U)$ is a finite $\mathbb{Z}_p$–algebra, B is also a finite $\mathbb{Z}_p$–algebra. Therefore it is a product of its localizations at finitely many ideals $B = \prod_{\mathfrak{m} \text{ max}} B_{\mathfrak{m}}$. The projection $Z_{\mathfrak{m}}$ of Z onto $B_{\mathfrak{m}}$ will either be contained in the maximal ideal $\mathfrak{m}$ of $B_{\mathfrak{m}}$ or contain a unit, we say that $\mathfrak{m}$ ***is ordinary*** if the latter holds. Let $B^{\mathrm{ord}} := \prod_{\mathfrak{m} \text{ ordinary}} B_{\mathfrak{m}}$. Each of these is a flat B–algebra and a subalgebra of $\mathrm{End}_{\mathbb{Z}_p}(U)$. Finally, we define $U^{\mathrm{ord}} := U \otimes_B B^{\mathrm{ord}}$, the ***ordinary part of*** U.

Now, taking Z to be Atkin's U–operator corresponding to p, we may consider the ordinary part of the homology group $H_1(Y_1(Np^r), \mathbb{Z}_p) = \Gamma_1(Np^r)^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p$. Notice that $(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$ is a Γ–module since the action of Γ commutes with U by (2.8).

(2.11) Proposition. If $r \geq s > 0$, then there is an isomorphism of abelian groups

$$(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s \xrightarrow{\sim} (\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}.$$

Proof. Take $U' : \Phi_{r-1}^{s,\mathrm{ab}} \to \Phi_r^{s\,\mathrm{ab}}$ as before. If $\pi : \Phi_r^{s\,\mathrm{ab}} \to \Phi_{r-1}^{s\,\mathrm{ab}}$ is the map induced by inclusion, then $U' \circ \pi = U \in \mathrm{End}(\Phi_r^{s\,\mathrm{ab}})$ and $\pi \circ U' = U \in \mathrm{End}(\Phi_{r-1}^{s\,\mathrm{ab}})$.

If $\mathfrak{m}$ is ordinary, then $U_{\mathfrak{m}}$ is a unit in $\mathrm{End}_{\mathbb{Z}_p}(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)_{\mathfrak{m}}$ so we can take $U^{-1} = \prod_{\mathfrak{m}\text{ ordinary}} U_{\mathfrak{m}}^{-1}$ acting on $(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z})^{\mathrm{ord}}$. Then π induces an isomorphism

$$(\Phi_r^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \cong (\Phi_{r-1}^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$$

with inverse $U^{-1} \circ U'$.

Inductively, we obtain

$$(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \cong (\Phi_s^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \cong (\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}.$$

Γ_s/Γ_r is p–torsion so $\Gamma_s/\Gamma_r \otimes \mathbb{Z}_p = \Gamma_s/\Gamma_r$. Also, U is Γ–equivariant so taking Γ_s–coinvariants (i.e. quotienting by $\mathfrak{a}_s$) and taking ordinary parts are commuting functors. Therefore, on tensoring by $\mathbb{Z}_p$ and taking ordinary parts in the sequence (4), we get:

$$1 \to (\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s \to (\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \to (\Gamma_s/\Gamma_r)^{\mathrm{ord}} \to 1.$$

By (2.9) U is nilpotent on Γ_s/Γ_r and so $(\Gamma_s/\Gamma_r)^{\mathrm{ord}} = 1$ which implies that

$$(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s \cong (\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}.$$

□

2.4 Iwasawa Modules

(2.12) Definition (Iwasawa module)**.** We may take a projective limit in the chain of $\mathbb{Z}_p$–modules

$$\cdots \to \Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p \to \cdots \to \Gamma_1(Np)^{\mathrm{ab}} \otimes \mathbb{Z}_p$$

to obtain

$$\mathbf{W} := \varprojlim_r \Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p.$$

The profinite group Γ acts on $\Gamma_1(Np^r) \otimes \mathbb{Z}_p$ through its finite quotient Γ/Γ_r and so $\mathbf{W}$ is a module over the completed group algebra

$$\Lambda := \varprojlim_r \mathbb{Z}_p[\Gamma/\Gamma_r].$$

(2.13) Fact. Suppose that M_r is a projective system of Λ–modules such that the M_r are invariant under Γ_r and such that for any $r \geq s$, the morphisms factor as $M_r \to M_r/\mathfrak{a}_s \to M_s$. Let $\mathbf{M} = \varprojlim_r M_r$ (so for any s, the natural morphism factors as $\mathbf{M} \to \mathbf{M}/\mathfrak{a}_s \to M_s$).

Suppose further that each M_r is p–adically complete and the morphisms $M_r/\mathfrak{a}_s \to M_s$ are isomorphisms. Then for any s, the morphism $\mathbf{M}/\mathfrak{a}_s \to M_s$ is an isomorphism.

In particular, this together with (2.11), implies that

(2.14) Theorem. For any $r \geq 1$, we have that the Γ_r–coinvariants of $\mathbf{W}^{\mathrm{ord}}$ are equal to $H_1(Y_r, \mathbb{Z}_p)^{\mathrm{ord}}$

$$\mathbf{W}^{\mathrm{ord}}/\mathfrak{a}_r \mathbf{W}^{\mathrm{ord}} = (\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}.$$

(2.15) Remark. Each module $\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p$ is free of finite rank over $\mathbb{Z}_p$ and so is compact in its p–adic topology. Therefore if we give $\mathbf{W}$ the projective limit topology arising from this it becomes a compact Λ–module, and Λ acts continuously on $\mathbf{W}$.

Since $\mathbf{W}^{\mathrm{ord}}$ is a direct factor of $\mathbf{W}$, this also holds for $\mathbf{W}^{\mathrm{ord}}$ and furthermore, (2.14) implies that the projective limit topology on $\mathbf{W}^{\mathrm{ord}}$ coincides with its $\mathfrak{m}$–adic topology (for $\mathfrak{m} = (\mathfrak{a}_1, p)$ the maximal ideal of Λ) because the kernels of $\Lambda \twoheadrightarrow \mathbb{Z}_p/p^r[\Gamma/\Gamma_r]$ are cofinal with the sequence of ideals $\mathfrak{m}^r$ in Λ.

Thus $\mathbf{W}^{\mathrm{ord}}$ is a Λ–module, compact in its $\mathfrak{m}$–adic topology, such that

$$\mathbf{W}^{\mathrm{ord}}/\mathfrak{m} = (\Gamma_1(Np)^{\mathrm{ab}} \otimes \mathbb{Z}_p/p)^{\mathrm{ord}}$$

is a finite dimensional $\mathbb{Z}_p/p$–module of dimension d. In particular, $\mathbf{W}^{\mathrm{ord}}$ is a finitely generated Λ–module with minimal generating set of order d, the $\mathbb{Z}_p$–rank of the free $\mathbb{Z}_p$–module $(\Gamma_1(Np)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$.

2.5 Main Theorems

Cohomology is the dual of homology:

$$H^1(Y_1(Np^r), \mathbb{Z}_p) := \mathrm{Hom}_{\mathbb{Z}}(\Gamma_1(Np^r)^{\mathrm{ab}}, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p)$$

and Λ acts on $\Gamma_1(Np^r) \otimes \mathbb{Z}_p$ through its quotient $\Lambda_r := \Lambda/\mathfrak{a}_r = \mathbb{Z}_p[\Gamma/\Gamma_r]$.

(2.16) Remark. If M is a $\mathbb{Z}_p[U]$–module finitely generated as a $\mathbb{Z}_p$–module, then the dual $M^* := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ is also finitely generated as a $\mathbb{Z}_p$–module and becomes a $\mathbb{Z}_p[U]$–module via the dual action of U. Further, $(M^*)^{\mathrm{ord}} = (M^{\mathrm{ord}})^*$.

(2.17) Remark. There is an isomorphism of Λ_r–modules[11]

$$\begin{aligned} \mathrm{Hom}_{\Lambda_r}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \Lambda_r) &= \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p[\Gamma/\Gamma_r] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p) \\ &\cong \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p). \end{aligned}$$

$\Lambda_s = \Lambda_r/\mathfrak{a}_s$ for $r \geq s > 0$, so we get a sequence of morphisms of Λ_r–modules

$$\begin{aligned} \mathrm{Hom}_{\Lambda_r}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \Lambda_r) &\to \mathrm{Hom}_{\Lambda_r}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \Lambda_r)/\mathfrak{a}_s \\ &\to \mathrm{Hom}_{\Lambda_r}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \Lambda_s) \\ &= \mathrm{Hom}_{\Lambda_s}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p/\mathfrak{a}_s, \Lambda_s). \end{aligned}$$

By (2.16), we may take ordinary parts to obtain

$$\begin{aligned} \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r) &\to \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r)/\mathfrak{a}_s \\ &\to \mathrm{Hom}_{\Lambda_s}((\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_s). \end{aligned}$$

(2.18) Lemma. The transfer $\Phi_r^{s\,\mathrm{ab}} \xrightarrow{V} \Gamma_1(Np^r)^{\mathrm{ab}}$ commutes with the actions of U.

[11] $\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{R[G]}(M, R[G] \otimes_R N)$ as right $R[G]$–modules for M, N left and right $R[G]$–modules respectively and for G a finite group, R a commutative ring.

Proof. The following diagram commutes[a]

$$\begin{array}{ccc}
\Phi_r^{s\,\mathrm{ab}} & \xrightarrow{\quad V \quad} & \Gamma_1(Np^r)^{\mathrm{ab}} \\
\big\downarrow V & & \big\downarrow V \\
(\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} & \xrightarrow{V} & (\Gamma_1(Np^r) \cap \Gamma^0(p))^{\mathrm{ab}} \\
\big\downarrow t(-)t^{-1} & & \big\downarrow t(-)t^{-1} \\
\Phi_r^{s\,\mathrm{ab}} & \xrightarrow{\quad V \quad} & \Gamma_1(Np^r)^{\mathrm{ab}}
\end{array}$$

□

[a] The commutativity of the top square follows from the functoriality of transfer. There are coset representatives for $\Gamma_1(Np^r) \cap \Gamma^0(p)$ in $\Phi_r^s \cap \Gamma^0(p)$ of the form $\sigma_d = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where d ranges through coset representatives of Γ_r in Γ_s. By the same argument as in (2.9), the conjugates $t\sigma_d t^{-1}$ form a set of coset representatives of $\Gamma_1(Np^r)$ in Φ_r^s, so the lower half commutes.

(2.19) Remark. Suppose G is a torsion-free congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then

$$\mathrm{Hom}_{\mathbb{Z}_p}(G^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}}(G^{\mathrm{ab}}, \mathbb{Z}_p) = H^1(Y(G), \mathbb{Z}_p)$$

where $Y(G) = G \setminus \mathfrak{h}$ an open Riemann surface. This curve can be completed to the compact Riemann surface $X(G)$ by the addition of finitely many points, the cusps. Intersection of cycles yields a canonical isomorphism

$$H^1(Y(G), \mathbb{Z}_p) \cong H_1(X(G), \mathrm{cusps}, \mathbb{Z}_p)$$

where the right hand is relative cohomology. The cusps correspond to the points of the orbit space $G \setminus \mathbb{P}^1(\mathbb{Q})$ and there is a canonical isomorphism

$$H_1(X(G), \mathrm{cusps}, \mathbb{Z}_p) \cong (\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \otimes \mathbb{Z}_p)/\mathfrak{a}_G$$

where $\mathfrak{a}_G$ is the augmentation ideal of $\mathbb{Z}[G]$ and $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ is made a $\mathbb{Z}[G]$–module via the action of G on $\mathbb{P}^1(\mathbb{Q})$.

It follows that if $H \leq G$ then there is the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{Z}_p}(H^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p) & \xrightarrow{V^*} & \mathrm{Hom}_{\mathbb{Z}_p}(G^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p) \\
\| & & \| \\
H^1(Y(H), \mathbb{Z}_p) & \longrightarrow & H^1(Y(G), \mathbb{Z}_p) \\
\| & & \| \\
H_1(X(H), \mathrm{cusps}, \mathbb{Z}_p) & \longrightarrow & H_1(X(G), \mathrm{cusps}, \mathbb{Z}_p) \\
\| & & \| \\
(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \otimes \mathbb{Z}_p)/\mathfrak{a}_H & \twoheadrightarrow & (\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \otimes \mathbb{Z}_p)/\mathfrak{a}_G
\end{array}$$

in which the horizontal maps are the dual of the transfer, the pushforward on cohomology, the pushforward on homology, and the natural quotient morphism.

This proves that V^* is surjective with kernel $\mathfrak{a}_G \operatorname{Hom}_{\mathbb{Z}_p}(H^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p)$.

(2.20) Lemma (A Control Lemma)**.** The morphism

$$\operatorname{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r)/\mathfrak{a}_s \to \operatorname{Hom}_{\Lambda_s}((\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_s)$$

in (2.17) is an isomorphism.

Proof. By (2.18) we may restrict V to the ordinary part to obtain a morphism

$$(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \xrightarrow{V} (\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$$

which induces a dual morphism

$$\operatorname{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p) \xrightarrow{V^*} \operatorname{Hom}_{\mathbb{Z}_p}((\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p).$$

The following diagram commutes

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p)) \arrow[r, "\sim"] \arrow[d, "V^*"] & \operatorname{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r) \arrow[d] \\
\operatorname{Hom}_{\mathbb{Z}_p}((\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p) \arrow[dd, equal] & \operatorname{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r)/\mathfrak{a}_s \arrow[d] \\
 & \operatorname{Hom}_{\Lambda_s}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s, \Lambda_s) \arrow[d, equal] \\
\operatorname{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p) \arrow[r, "\sim"] & \operatorname{Hom}_{\Lambda_s}((\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \Lambda_s)^{\mathrm{ord}}, \Lambda_s)
\end{tikzcd}$$

where the equalities follow from (2.11) and its proof.

Thus to prove the lemma it suffices to prove that

$$\operatorname{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p) \xrightarrow{V^*} \operatorname{Hom}_{\mathbb{Z}_p}((\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p) \tag{5}$$

is surjective with kernel $\mathfrak{a}_s \operatorname{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p)$. But by (2.18) and (2.16), (5) is the ordinary part of the morphism

$$\operatorname{Hom}_{\mathbb{Z}_p}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{V^*} \operatorname{Hom}_{\mathbb{Z}_p}(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p) \tag{6}$$

and taking ordinary parts is exact and commutes with the action of Γ so it suffices to show instead that (6) is surjective with kernel $\mathfrak{a}_s \operatorname{Hom}_{\mathbb{Z}_p}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p)$. Taking $H = \Gamma_1(Np^r)$ and $G = \Phi_r^s$ in (2.19) proves this. □

We now prove the following key theorem of Hida (2.22), assuming the following fact (which holds due to a result by Serre [9] as Λ is a regular local ring of dimension two):

(2.21) Fact. Any finitely generated reflexive Λ–module is free.

(2.22) Theorem. The Λ–module $\mathbf{W}^{\mathrm{ord}}$ is free of finite rank d.

Proof. By (2.14),

$$\begin{aligned}\mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}},\Lambda) &= \varprojlim_r \mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}},\Lambda_r)\\ &= \varprojlim_r \mathrm{Hom}_{\Lambda_r}(\mathbf{W}^{\mathrm{ord}}/\mathfrak{a}_r,\Lambda_r)\\ &= \varprojlim_r \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_r)\end{aligned}$$

and then, using (2.13) and (2.20), there is a canonical isomorphism

$$\mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}},\Lambda)/\mathfrak{a}_r = \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}}. \tag{7}$$

We have the following series of canonical isomorphisms, showing that $\mathbf{W}^{\mathrm{ord}}$ is a reflexive Λ–module:

$$\begin{aligned}\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}},\Lambda),\Lambda) &= \varprojlim_r \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}},\Lambda),\Lambda_r)\\ &= \varprojlim_r \mathrm{Hom}_{\Lambda_r}(\mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}},\Lambda)/\mathfrak{a}_r,\Lambda_r)\\ &= \varprojlim_r \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_r),\Lambda_r) &&(8)\\ &= \varprojlim_r (\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}} &&(9)\\ &= \mathbf{W}^{\mathrm{ord}}\end{aligned}$$

where (8) follows from (7), and (9) follows from the fact that $(\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}}$ is a reflexive $\mathbb{Z}_p$–module as it is a free $\mathbb{Z}_p$–module, and is therefore a reflexive Λ_r–module[a].

We can then apply (2.21) and are done. □

[a]If M is a left $R[G]$–module for R a commutative ring and G a commutative group, if M is reflexive as an R–module, it is reflexive as an $R[G]$–module.

3 Emerton's Ordinary Parts Functor

In 2010, Emerton published the paper [5] in which he constructed the functor Ord_P and demonstrated a few of its basic properties, such as being left exact and additive, commuting with inductive limits, and preserving admissibility. Ord_P provides a generalisation of the notion of ordinary parts described previously, and in particular allows the representation theoretic approach taken in section 2 (applying it to p–adically completed cohomology) to be generalised. For instance, (3.24) abstracts Hida's principle that the ordinary part of cohomology should be finite over the weight space.

Beyond this, the ordinary parts functor can be used in the construction of the the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (c.f. [1] and[4]). In particular, one can derive it, which allows us to compute the vanishing of certain Ext groups between irreducible admissible representations (c.f. [6]).

An important property of Ord_P is that, for P a parabolic subgroup of a p–adic reductive group G with Levi decompositon $P = MN$ and opposite parabolic $\overline{P}$, the functor $\mathrm{Ord}_P : \mathrm{Mod}_G^{\mathrm{adm}}(A) \to \mathrm{Mod}_M^{\mathrm{adm}}(A)$ is right adjoint to parabolic induction $\mathrm{Ind}_{\overline{P}}^G : \mathrm{Mod}_M^{\mathrm{adm}}(A) \to \mathrm{Mod}_G^{\mathrm{adm}}(A)$. This suggests that it is analogous to the locally analytic Jacquet functor J_P and indeed many of the arguments are the same as those appearing in Emerton's previous papers [3] and [7] studying this functor.

We shall, following [5], begin by defining certain categories of p–adic representations of G. We will then define Ord_P, and finally prove that it is right adjoint to $\mathrm{Ind}_{\overline{P}}^G$.

3.1 p–adic Representations

Let G be a p–adic analytic group, in our case the group of $\mathbb{Q}_p$–points of a connected reductive algebraic group over $\mathbb{Q}_p$. Let A be a complete local Noetherian $\mathcal{O}$–algebra for $\mathcal{O}$ the ring of integers of some finite $E/\mathbb{Q}_p$, with a choice of uniformiser ϖ. Further, let A has finite residue field and maximal ideal $\mathfrak{m}$. Let $\mathrm{Mod}_G(A)$ denote the category of $A[G]$–modules.

In this subsection we will define certain categories of G–representations and outline their basic properties.

(3.1) Definition (Completed group ring)**.** If $H \leq G$ is compact open, let $A[[H]]$ denote the ***completed group ring*** of H over A

$$A[[H]] := \varprojlim_{H' \leq H \text{ open}} A[H/H'].$$

Each $A[H/H']$ is endowed with the $\mathfrak{m}$-adic topology and $A[[H]]$ has the projective limit topology arising from this.

Each $A[H/H']$ is profinite as H/H' is finite (as H is compact) and so $A[[H]]$ is profinite, so is in particular a compact topological ring.

(3.2) Definition (Smooth representations)**.** If V is a representation, $v \in V$ is ***smooth*** if it is annihilated by an open ideal in $A[[H]]$ for some (equiv. any) open $H \leq G$. Equivalently, if it is fixed by some open subgoup of G and if $\mathfrak{m}^i v = 0$ for some i, where $\mathfrak{m}$ is the maximal ideal of A[12]. Let V_{sm} be the set of smooth vectors in V. V is a ***smooth representation*** if $V = V_{\mathrm{sm}}$.

Let $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G(A)$ consisting of smooth G-representations. This is an abelian category.

(3.3) Definition (Admissible representations)**.** $V \in \mathrm{Mod}_G^{\mathrm{sm}}(A)$ is ***admissible*** if $V^H[\mathfrak{m}^i]$ (the $\mathfrak{m}^i$–torsion part of V^H) is finitely generated over A for every $H \leq G$ open and $i \geq 0$. (For A Artinian, $\mathfrak{m}^i = 0$ for large i so $V^H[\mathfrak{m}^i] = V^H$ and this agrees with the usual definition.)

Let $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ consisting of admissible representations. This is an abelian category.

(3.4) Definition (Locally admissible representations)**.** Let $V \in \mathrm{Mod}_G(A)$. $v \in V$ is ***locally admissible*** if v is smooth and if the smooth G–subrepresentation of V generated by v is admissible. Let $V_{\mathrm{l.adm}}$ denote the subset of locally admissible elements, an $A[G]$–submodule of V.

[12] If A is Artinian, then $\mathfrak{m}^i = 0$ for some i so we can drop the second condition.

V is ***locally admissible*** if $V = V_{\text{l.adm}}$ and the category of these is denoted $\mathrm{Mod}_G^{\text{l.adm}}(A)$. This is an abelian category.

(3.5) Definition (Finiteness)**.** Let $V \in \mathrm{Mod}_G(A)$ and Z be the centre of G. We say that V is Z***–finite*** if $A[Z]/\,\mathrm{Ann}_{A[Z]}(V)$ is a finite A–algebra.

$v \in V$ is ***locally*** Z***–finite*** if the $A[Z]$–submodule of V generated by v is Z–finite (equivalently finitely generated as an A–module). Let $V_{Z\text{–fin}} \subset V$ denote the set of locally Z–finite elements. This is an $A[G]$–submodule of V.

(3.6) Remark. The set of Z–finite (resp. locally Z–finite) modules is closed under taking direct sums and quotients by Z–finite (resp. locally Z–finite) modules. It follows that if $V \in \mathrm{Mod}_G(A)$ is finitely generated over $A[G]$, then V is Z–finite iff it is locally Z–finite.

Also, we can show that for any $V \in \mathrm{Mod}_G(A)$, $V_{\text{l.adm}} \subset V_{Z\text{–fin}}$, so in particular if V is admissible and finitely generated, then it is Z–finite.

(3.7) Definition (ϖ–adic continuity)**.** We say that $V \in \mathrm{Mod}_G(A)$ is ϖ***–adically continuous*** if

1. V is ϖ–adically separated and complete.
2. The $\mathcal{O}$–torsion subspace $V[\varpi^\infty]$ of V is of bounded exponent (i.e. is annihilated by ϖ^j for large j).
3. The G–action $G \times V \to V$ is continuous in the ϖ–adic topology on V.
4. The A–action $A \times V \to V$ is continuous in the $\mathfrak{m}$–adic topology on A and the ϖ–adic topology on V.

Let $\mathrm{Mod}_G^{\varpi\text{-cont}}(A)$ denote the full subcategory of $\mathrm{Mod}_G(A)$ consisting of ϖ–adically continuous representations of G over A. It is closed under taking kernels, images, and extensions.

(3.8) Definition. We say $V \in \mathrm{Mod}_G^{\varpi\text{-cont}}(A)$ is ***admissible*** if (with the induced G–representation) $(V/\varpi V)[\mathfrak{m}] \in \mathrm{Mod}_G^{\text{adm}}(A/\mathfrak{m})$.

Let $\mathrm{Mod}_G^{\varpi\text{-adm}}(A)$ denote the category consisting of these. This is an abelian category.

(3.9) Remark. When $A \in \mathrm{Art}(\mathcal{O})$, $\mathrm{Mod}_G^{\varpi\text{-cont}}(A) = \mathrm{Mod}_G^{\text{sm}}(A)$ and $\mathrm{Mod}_G^{\varpi\text{-adm}}(A) = \mathrm{Mod}_G^{\text{adm}}(A)$.

3.2 The Ordinary Parts Functor

We will now define the Hecke action and construct the ordinary parts functor.

Let P be parabolic in G with Levi decomposition $P = MN$. The example to have in mind is $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with maximal compact $\mathrm{GL}_2(\mathbb{Z}_p)$, P the upper triangular matrices, N upper triangular matrices with 1's on the diagonal, and M the diagonal matrices.

(3.10) Definition. If $V \in \mathrm{Mod}_N(A)$ and $N_1 \subset N_2 \subset N$ are compact open subgroups, define the operator $h_{N_2,N_1} : V^{N_1} \to V^{N_2}$ by

$$h_{N_2,N_1}(v) := \sum_{n \in N_2/N_1} nv,$$

summing over coset representatives. It is easy to check that this is well defined and that for $N_1 \subset N_2 \subset N_3$, $h_{N_3,N_2} h_{N_2,N_1} = h_{N_3,N_1}$.

(3.11) Notation. Let $P_0 \leq P$ be compact open and set $M_0 = M \cap P_0$, $N_0 = N \cap P_0$,

$$M^+ := \{m \in M \mid mN_0m^{-1} \subset N_0\}.$$

Let Z_M be the centre of M, and set $Z_M^+ := M^+ \cap Z_M$.

(3.12) Definition (Hecke action)**.** Let $V \in \mathrm{Mod}_P(A)$. Then for any $m \in M^+$ we define $h_{N_0,m} : V^{N_0} \to V^{N_0}$ via

$$h_{N_0,m}(v) := h_{N_0,mN_0m^{-1}}(mv).$$

Note that if $m_1, m_2 \in M^+$, then[13] $h_{N_0,m_1m_2} = h_{N_0,m_1}h_{N_0,m_2}$, so $h_{N_0,m}$ induces an action of M^+ on V^{N_0}, which we call the ***Hecke action of*** M^+.

This allows us to regard V^{N_0} as a module over the algebra $A[M^+]$, and over its central subalgebra $A[Z_M^+]$. If $m \in M_0$, then $mN_0m^{-1} = N_0$ so $h_{N_0,m}$ coincides with the action of m on V^{N_0}.

With this action, we may also consider the $A[Z_M]$–module $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})$ as well as its submodule $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})_{Z_M\text{-fin}}$.

We assume following result from [3]:

(3.13) Proposition. Let N_i, N_i' be compact subgroups of N for $1 \leq i \leq n$.

1. If $Y = \{z \in Z_M \mid zN_iz^{-1} \subset N_i' \,\forall\, 1 \leq i \leq n\}$ then Y generates Z_M as a group.
2. If $Y' = \{m \in M \mid mN_im^{-1} \subset N_i' \,\forall\, 1 \leq i \leq n\}$ then Y' and Z_M together generate M as a semigroup.

In particular Y' generates M as a group.

(3.14) Corollary. Z_M is generated by Z_M^+ as an abelian group.

(3.15) Definition (Ordinary parts functor)**.** If $V \in \mathrm{Mod}_P^{\mathrm{sm}}(A)$, then we write

$$\mathrm{Ord}_P(V) := \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})$$

which we refer to as the P–***ordinary part of*** V.

Assume temporarily that Ord_P is well defined independent of the choice of P_0 and M.

(3.16) Remark. To relate this to the definition (2.10) in section 2, suppose U is an $A[Z_M]$–module and finitely generated over A. Then $B = \mathrm{Im}(A[Z_M^+] \to \mathrm{End}_A(U)) \cong \prod_{\mathfrak{m}\text{ max}} B_{\mathfrak{m}}$ induces a factorization $U = \prod_{\mathfrak{m}\text{ max}} U_{\mathfrak{m}}$, and so a factorization

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U) \cong \prod_{\mathfrak{m}\text{ max}} \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U_{\mathfrak{m}}).$$

If $\mathfrak{m}$ is ordinary, the $A[Z_M^+]$–module structure on $U_{\mathfrak{m}}$ extends uniquely to an $A[Z_M]$–module structure and evaluation at $1 \in Z_M$ induces an isomorphism $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U_{\mathfrak{m}}) \cong U_{\mathfrak{m}}$. If $\mathfrak{m}$ is non-ordinary, then $\mathrm{Hom}_{A[Z_M^+]}(Z_M, U_{\mathfrak{m}}) = 0$.

Thus, evaluation at 1 induces an embedding

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U) \cong \prod_{\mathfrak{m}\text{ ordinary}} U_{\mathfrak{m}} \subset U.$$

Notice that this also shows that $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U)$ is finitely generated over A.

[13] $h_{N_0,m_1m_2} = h_{N_0,m_1m_2N_0m_2^{-1}m_1^{-1}}m_1m_2 = h_{N_0,m_1N_0m_1^{-1}}h_{m_1N_0m_1,m_1m_2N_0m_2^{-1}m_1^{-1}}m_1m_2 = h_{N_0,m_1N_0m_1^{-1}}m_1h_{N_0,m_2N_0m_2^{-1}}m_2 = h_{N_0,m_1}h_{N_0,m_2}$.

(3.17) Proposition. Ord_P is a functor $\mathrm{Mod}_P^{\mathrm{sm}}(A) \to \mathrm{Mod}_M^{\mathrm{sm}}(A)$.

Proof. Let W be an $A[Z_M^+]$–module and take $\phi \in \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M\text{-fin}}$.

Let $\mathrm{ev}_z : \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M\text{-fin}} \to W$ denote evaluation at $z \in Z_M^+$ and consider the $A[Z_M]$–submodule $U = \langle \phi \rangle \subset \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M\text{-fin}}$.

$ev_1(U) = \mathrm{im}(\phi)$ and since ϕ is locally Z_M–finite, U is finitely generated over A and so $\mathrm{im}(\phi)$ also is. Since $z_1\phi(z_2) = \phi(z_1 z_2)$ for $z_1 \in Z_M^+$, $z_2 \in Z_M$, $\mathrm{im}(\phi)$ is invariant under the action of Z_M^+.

Now suppose that W is additionally an $A[M^+]$–module.

There is an isomorphism

$$\begin{aligned} M^+ \times_{Z_M^+} Z_M := M^+ \times Z_M / ((mz^+, z) \sim (m, z^+ z)) &\xrightarrow{\sim} M \\ (m, z) &\mapsto mz \end{aligned}$$

which is a bijection[a].

The restriction map induces an isomorphism

$$\mathrm{Hom}_{A[M^+]}(A[M], W) \cong \mathrm{Hom}_{A[M^+]}(A[M^+ \times_{Z_M^+} Z_M], W) \xrightarrow{\sim} \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)$$

which, due to the natural $A[M]$–structure on $\mathrm{Hom}_{A[M^+]}(A[M], W)$, allows us to endow $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)$ with an $A[M]$–structure which extends its original $A[Z_M]$–module structure. This has the property that for any $z \in Z_M$, ev_z is M^+–equivariant.

Let us further assume that W is M_0–smooth.

Since $\mathrm{im}(\phi)$ is a finitely generated A–submodule of W, we may find an open ideal I of $A[[M_0]]$ that annihilates $\mathrm{im}(\phi)$, so I annihilates ϕ also. Thus the action of M_0 on $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M\text{-fin}}$ is smooth, so the M action is also.

V^{N_0} with the Hecke action satisfies the properties assumed for W and the action of M_0 is smooth, so the above holds with $W = V^{N_0}$. Therefore $\mathrm{Ord}_P(V)$ is a smooth M–representation and we are done. $\square$

[a] It is injective as, if $m_1 z_1 = m_2 z_2$, then as Z_M^+ generates Z_M there is $z \in Z_M^+$ such that $z_1 z_2^{-1} z \in Z_M^+$ so $(m_1, z_1) \sim (m_1 z_1 z_2^{-1} z, z^{-1} z_2) = (m_2 z, z^{-1} z_2) \sim (m_2, z_2)$. It is surjective as M^+ and Z_M generate M together by (3.13).

(3.18) Proposition. Ord_P is well defined.

Proof. We want to prove first that Ord_P is independent of the choice of P_0.

Suppose that P_0' is an open subgroup of P_0 and write $M_0' := M \cap P_0'$, $N_0' = N \cap P_0'$, $(M^+)' = \{m \in M \mid mN_0' m^{-1} \subset N_0\}$, $(Z_M)' = (M^+)' \cap Z_M$. We can define the Hecke action of $(M^+)'$ on $V_0^{N_0'}$ in the obvious way.

Write $Y' = M^+ \cap (M^+)'$ and $Y = Z_M^+ \cap (Z_M^+)' = Y' \cap Z_M$. Then by (3.13), Y generates

Z_M as a group so

$$\mathrm{Hom}_{A[(Z_M^+)']}(A[Z_M], V^{N_0'}) = \mathrm{Hom}_{A[Y]}(A[Z_M], V^{N_0'})$$
$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0}) = \mathrm{Hom}_{A[Y]}(A[Z_M], V^{N_0})$$

so we have a map

$$\phi : \mathrm{Hom}_{A[(Z_M^+)']}(A[Z_M], V^{N_0'}) \xrightarrow{f \mapsto h_{N_0,N_0'} \circ f} \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0}).$$

If $m \in Y'$ then[a] $h_{N_0,m} h_{N_0,N_0'} = h_{N_0,N_0'} h_{N_0',m}$, so $h_{N_0,N_0'}$ commutes with the Hecke action of Y' on V^{N_0} and $V^{N_0'}$. By (3.13), Y' and Z_M together generate M, so ϕ is M–equivariant.

For $z \in Y$ with $zN_0z^{-1} \subset N_0'$, define

$$\alpha : V^{N_0} \xrightarrow{v \mapsto h_{N_0', zN_0z^{-1}}(zv).} V^{N_0'}.$$

We can check that[b] $h_{N_0', zN_0z^{-1}} z h_{N_0,m} = h_{N_0,m} h_{N_0', zN_0z^{-1}} z$ so the action commutes with the Hecke actions of Y', and so there is an M–equivariant map

$$\psi : \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0}) \xrightarrow{f \mapsto \alpha \circ f} \mathrm{Hom}_{A[(Z_M^+)']}(A[Z_M], V^{N_0'}).$$

$h_{N_0,N_0'} h_{N_0', zN_0z^{-1}} z = h_{N_0,z}$ and[c] $h_{N_0', zN_0z^{-1}} z h_{N_0,N_0'} = h_{N_0,z}$, so the composites $\phi\psi$ and $\psi\phi$ coincide with the automorphisms induced by composing with the Hecke actions of z. Therefore $\phi\psi$ and $\psi\phi$ are both isomorphisms, and so ϕ is both injective and surjective, so ϕ is an isomorphism.

This concludes the proof that Ord_P is well defined up to canonical isomorphism independently of the choice of P_0. It remains to show that it is also independent of the choice of Levi factor M.

Suppose that M' is another Levi factor. There are canonical isomorphisms $M \cong P/N \cong M'$ and there is a unique $n \in N$ with $nMn^{-1} = M'$. Conjugation by n is an isomorphism, so we can regard any M'–representation as an M–representation and vice versa.

Pick $P_0 \subset P$ compact open and define $P_0' = nP_0n^{-1}$, $M_0' = nM_0n^{-1} = nMn^{-1} \cap P_0'$, $N_0' = nN_0n^{-1} = N \cap P_0'$, $(M')^+ = nM^+n^{-1} = \{m' \in M' \mid m'N_0'(m')^{-1} \subset N_0'\}$, $Z_{M'}^+ = nZ_M^+n^{-1} = (M')^+ \cap Z_{M'}$.

We define a Hecke action of $(M')^+$ on $V^{N_0'}$ in the obvious way and can show as before that $\mathrm{Hom}_{A[Z_M^+]}(A[Z_{M'}], V^{N_0'})$ is an M'–representation. Therefore $\mathrm{Hom}_{A[Z_M^+]}(A[Z_{M'}], V^{N_0'})_{Z_M\text{-fin}}$ also is and so it is an M–representation.

Define the functor

$$\begin{aligned} \mathrm{Ord}_P' : \mathrm{Mod}_G(A) &\to \mathrm{Mod}_M(A) \\ V &\mapsto \mathrm{Hom}_{A[Z_{M'}^+]}(A[Z_{M'}], V^{N_0'})_{Z_M\text{-fin}}. \end{aligned}$$

There is an isomorphism $V^{N_0} \xrightarrow{v \mapsto nv} V^{N_0'}$ which commutes[d] with the Hecke operator h_m on V^{N_0} and the operator $h_{nmn^{-1}}$ on $V^{N_0'}$ for all $m \in M^+$. So there is an M–equivariant isomomorphism

$$\mathrm{Ord}_P(V) = \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0}) \xrightarrow{\sim} \mathrm{Hom}_{A[Z_{M'}^+]}(A[Z_{M'}], V^{N_0'}) = \mathrm{Ord}'_P(V)$$
$$f \mapsto nf.$$

So Ord_P is well defined up to canonical isomorphism independently of M. □

[a] $h_{N_0,m} h_{N_0,N_0'} = h_{N_0,mN_0m^{-1}} m h_{N_0,N_0'} = h_{N_0,mN_0m^{-1}} h_{mN_0m^{-1},mN_0'm^{-1}} m = h_{N_0,mN_0'm^{-1}} m = h_{N_0,N_0'} h_{N_0',mN_0'm^{-1}} m = h_{N_0,N_0'} h_{N_0',m}$.

[b] Using that $z \in Z_M$, $h_{N_0',zN_0z^{-1}} z h_{N_0,m} = h_{N_0',zN_0z^{-1}} z h_{N_0,mN_0m^{-1}} m = h_{N_0',zN_0z^{-1}} h_{zN_0z^{-1},zmN_0m^{-1}z^{-1}} zm = h_{N_0',zmN_0m^{-1}z^{-1}} zm = h_{N_0',mN_0'm^{-1}} h_{mN_0'm^{-1},mzN_0z^{-1}m^{-1}} mz = h_{N_0',mN_0'm^{-1}} m h_{N_0',zN_0z^{-1}} z = h_{N_0,m} h_{N_0',zN_0z^{-1}} z$.

[c] $h_{N_0',zN_0z^{-1}} z h_{N_0,N_0'} = h_{N_0',zN_0,z^{-1}} h_{zN_0z^{-1},zN_0'z^{-1}} z = h_{N_0',zN_0'z^{-1}} z = h_{N_0,z}$.

[d] $nh_m = nh_{N_0,mN_0m^{-1}} m = h_{nN_0n^{-1},nmN_0m^{-1}n^{-1}} nm = h_{N_0',nmn^{-1}N_0'nm^{-1}n^{-1}} nmn^{-1}n = h_{nmn^{-1}} n$.

(3.19) Definition. Evaluation at $1 \in A[Z_M]$ induces $\mathrm{Ord}_P(V) \to V^{N_0}$, the ***canonical lifting***.

(3.20) Facts (Properties of Ord_P).

- If $\{W_i\}_{i\in I}$ is an inductive system of smooth Z_M^+–modules, then the natural map is an isomorphism
$$\varinjlim_i \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W_i)_{Z_M\text{-fin}} \xrightarrow{\sim} \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], \varinjlim_i W_i)_{Z_M\text{-fin}}.$$
- $\mathrm{Ord}_P : \mathrm{Mod}_P(A) \to \mathrm{Mod}_M(A)$ is left exact, additive, and commutes with inductive limits.

(3.21) Fact. Let I_0, I_2 be compact open subgroups of G admitting Iwahori decompositions with respect to P and $\overline{P}$. Suppose further that $I_1 \cap M \subset I_0 \cap \overline{N}$, $I_1 \cap M = I_0 \cap M$, and $I_1 \cap N = I_0 \cap N = N_0$. Then if $z_0 \in Z_M^+$ is such that $(I_0 \cap \overline{N}) \subset z_0(I_1 \cap \overline{N})z_0^{-1}$, then $h_{N_0,z_0}(V^{I_1}) \subset V^{I_0}$.

(3.22) Theorem. If V is an admissible smooth representation of G over A, then $\mathrm{Ord}_P(V)$ is an admissible smooth representation of M and the canonical lifting is an embedding.

Sketch Proof. V is smooth, so it is an inductive limit of its submodules $V[\mathfrak{m}^i]$. As Ord_P is left exact and preserves inductive limits, $\mathrm{Ord}_P(V[\mathfrak{m}^i]) \hookrightarrow \mathrm{Ord}_P(V)[\mathfrak{m}^i]$ is an isomorphism and $\mathrm{Ord}_P(V)$ is an inductive limit of $\mathrm{Ord}_P(V)[\mathfrak{m}^i]$. So it suffices to prove the theorem for $V[\mathfrak{m}^i]$ instead of V.

Choose a cofinal sequence $\{I_i\}_{i\geq 0}$ of compact open subgroups of G with I_i normal in I_0 and admitting an Iwahori decomposition[a] with respect to P and $\overline{P}$. Let $M_i = I_i \cap M$, $N_i = I_i \cap N$, $\overline{N}_i = I_i \cap \overline{N}$. Then for $i \geq j$, $I_{i,j} = \overline{N}_i M_j N_0$ is compact open in I_0 and has an Iwahori decomposition. It is enough show that $\mathrm{Ord}_P(V)^{M_j}$ is finitely generated over A and the canonical lifting induces an embedding $\mathrm{Ord}_P(V)^{M_j} \hookrightarrow V^{N_0}$ for all $j \geq 0$.

Let $P_0 = M_0 N_0$, a compact subgroup of P. $V^{M_jN_0}$ is invariant under the Hecke Z_M^+–action on V^{N_0} so $\mathrm{Ord}_P(V)^{M_j} = \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{M_jN_0})_{Z_M\text{-fin}}$. For $\phi \in \mathrm{Ord}_P(V)^{M_j}$, $\mathrm{im}(\phi)$ is a finitely generated Z_M^+–invariant A–module, so $\mathrm{im}(\phi) \subset V^{I_{i,j}}$ for large i, j.

If we take $z_0 \in Z_M^+$ with $\overline{N}_j \subset z_0 \overline{N}_i z_0^{-1}$, then[b] $\mathrm{im}(\phi) = h_{N_0,z_0}\,\mathrm{im}(\phi) \subset h_{N_0,z_0}(V^{I_{i,j}}) \subset V^{I_{i,j}}$. So if U denotes the maximal A–submodule of $V^{I_{i,j}}$ invariant under the Hecke Z_M^+–

action, then $\mathrm{im}(\phi) \subset U$ and consequently $\mathrm{Ord}_P(V)^{M_j} \subset \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U)$, and so is finitely generated over A by (3.16). (3.16) also shows that the canonical lifting induces an embedding $\mathrm{Ord}_P(V)^{M_j} \hookrightarrow U \subset V^{I_{i,j}} \subset V^{N_0}$. □

[a] $(I \cap \overline{N}) \times (I \cap M) \times (I \cap N) \to I$ is a bijection.
[b] If $z \in Z_M$, then $\phi(z) = \phi(z_0 z_0^{-1} z) = h_{N_0,z_0}\phi(z_0^{-1}z) \in h_{N_0,z_0}(V^{I_{i,j}})$.
For the last inequality, we use (3.21).

(3.23) Definition (Extension to ϖ–adically complete representations)**.** If $V \in \mathrm{Mod}_P^{\varpi\text{-cont}}(A)$, then we define

$$\mathrm{Ord}_P(V) := \varprojlim_i \mathrm{Ord}_P((V/\varpi^i V))$$

and this is well defined as each $V/\varpi^i V \in \mathrm{Mod}_P^{\mathrm{sm}}(A/\mathfrak{m}^j)$.

This defines a functor

$$\mathrm{Ord}_P : \mathrm{Mod}_P^{\varpi\text{-cont}}(A) \to \mathrm{Mod}_M^{\varpi\text{-cont}}(A)$$

and we can show that for sufficiently large j, there is an embedding $\mathrm{Ord}_P(V)/\varpi^j \mathrm{Ord}_P(V) \to \mathrm{Ord}_P(V/\varpi^j V)$ and the projective limit topology on $\mathrm{Ord}_P(V)$ coincides with the ϖ–adic topology on $\mathrm{Ord}_P(V)$.

We have the canonical lifting $\mathrm{Ord}_P(V/\varpi^i V) \to (V/\varpi^i V)^{N_0}$, so passing to projective limits gives an M^+–equivariant map $\mathrm{Ord}_P(V) \to V^{N_0}$ which we again refer to as the canonical lifting.

(3.24) Theorem. Taking ordinary parts induces a functor $\mathrm{Ord}_P : \mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A) \to Mod_M^{\varpi-\mathrm{adm}}(A)$. Furthermore, for any $V \in \mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, the canonical lifting above is a closed embedding when its source and target have their ϖ–adic topologies.

3.3 Parabolic Induction

(3.25) Definition (Parabolic induction)**.** Let $\overline{P}$ be the opposite parabolic to P with unipotent radical $\overline{N}$ and $M = P \cap \overline{P}$.

If $U \in \mathrm{Mod}_M^{\mathrm{sm}}(A)$ then we can regard it as a P–representation by letting N act as the identity. Define

$$\mathrm{Ind}_{\overline{P}}^G U = \{f : G \to U \mid f \text{ locally constant}, f(\bar{p}g) = \bar{p}f(g)\, \forall \bar{p} \in \overline{P}, g \in G\}.$$

If $U \in \mathrm{Mod}_M^{\varpi\text{-cont}}(A)$, then we can again regard it as a P–representation and define

$$\mathrm{Ind}_{\overline{P}}^G U = \{f : G \to U \mid f \text{ cts } (U \text{ with } \varpi\text{–adic topology}), f(\bar{p}g) = \bar{p}f(g)\, \forall \bar{p} \in \overline{P}, g \in G\}.$$

The right regular action of G on functions gives these the structure of $A[G]$–modules.

(3.26) Remark. Parabolic induction $U \mapsto \mathrm{Ind}_{\overline{P}}^G U$ gives rise to exact functors $\mathrm{Mod}_M^{\mathrm{sm}}(A) \to \mathrm{Mod}_G^{\mathrm{sm}}(A)$ and $\mathrm{Mod}_M^{\varpi\text{-cont}}(A) \to \mathrm{Mod}_G^{\varpi\text{-cont}}(A)$.

It also respectively maps admissible smooth, locally admissible smooth, and ϖ–adically admissible $A[M]$–modules to admissible smooth, locally admissible smooth, and ϖ–adically admissible $A[G]$–modules.

(3.27) Remark. For $U \in \mathrm{Mod}_{\overline{P}}^{G}(A)$, let $\mathcal{C}^{\mathrm{sm}}(\overline{P} \setminus G, U) = \{\overline{P} \setminus G \to U \text{ locally constant}\}$. We can pullback along a section σ of $G \twoheadrightarrow \overline{P} \setminus G$ (this exists since G is a locally trivial $\overline{P}$–bundle over $\overline{P} \setminus G$) to get an A–linear isomorphism $\mathrm{Ind}_{\overline{P}}^{G} U \cong \mathcal{C}^{\mathrm{sm}}(\overline{P} \setminus G, U)$.

If $U \in \mathrm{Mod}_{M}^{\mathrm{sm}}(A)$ and $f \in \mathrm{Ind}_{\overline{P}}^{G} U$, then f is locally constant and $\mathrm{supp}(f)$ is an open and closed subset of G which is invariant under left translation by $\overline{P}$, so it can be considered as a compact open subset of $\overline{P} \setminus G$. There is an open immersion $N \hookrightarrow \overline{P} \setminus G$ so we can also regard N as an open subset of $\overline{P} \setminus G$.

Consider the A–submodule $(\mathrm{Ind}_{\overline{P}}^{G} U)(N) = \{f \in \mathrm{Ind}_{\overline{P}}^{G} U \mid \mathrm{supp}(f) \subset N\}$. For such f, $f|_N$ is a locally constant, compactly supported functor $N \to U$, so there is an isomorphism[14] of A–modules $(\mathrm{Ind}_{\overline{P}}^{G} U)(N) \to \mathcal{C}_c^{\mathrm{sm}}(N, U)$. This is locally constant since f is compactly supported and is an element of $\mathrm{Ind}_{\overline{P}}^{G} U(N)$ mapping to f.

N is invariant under the translation action of P on $\overline{P} \setminus G$ ($n' \cdot mn = m^{-1}n'mn$), so $(\mathrm{Ind}_{\overline{P}}^{G} U)(N)$ is P–invariant. This gives a P–action on $\mathcal{C}_c^{\mathrm{sm}}(N, U)$ ($(mnf)(n') = mf(m^{-1}n'mn)$), and there is an isomorphism of $A[P]$–modules $\mathcal{C}_c^{\mathrm{sm}}(N, A) \otimes_A U \cong \mathcal{C}_c^{\mathrm{sm}}(N, U)$.

(3.28) Lemma. Let $V \in \mathrm{Mod}_{P}^{\mathrm{sm}}(A)$. The map $\mathrm{ev}_{1_{N_0}} : \mathrm{Hom}_{A[N]}(\mathcal{C}_c^{\mathrm{sm}}(N, A), V) \to V^{N_0}$ induced by evaluation at the indicator function $1_{N_0} \in \mathcal{C}_c^{\mathrm{sm}}(N, A)$ is M^+–equivariant if we equip V^{N_0} with its Hecke M^+–action.

Further, as $\mathrm{Hom}_{A[N]}(\mathcal{C}_c^{\mathrm{sm}}(N, A), V)$ is an M–module (as M normalises N), this induces an M–equivariant isomorphism

$$\mathrm{Hom}_{A[N]}(\mathcal{C}_c^{\mathrm{sm}}(N, A), V) \to \mathrm{Hom}_{A[M^+]}(A[M], V^{N_0}) \cong \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0}). \tag{10}$$

Proof. The first part is simple.

Let $\phi \in \mathrm{Hom}_{A[N]}(\mathcal{C}_c^{\mathrm{sm}}(N, A), V)$ map to $\tilde{\phi} \in \mathrm{Hom}_{A[M^+]}(A[M], V^{N_0})$ under the above isomorphism.

If $m \in M^+$, then $\tilde{\phi}(m) = (m\tilde{\phi})(1) = (m\phi)(1_{N_0}) = \phi(m1_{N_0}) = \phi(1_{mN_0m^{-1}})$.

Since any element of $\mathcal{C}_c^{\mathrm{sm}}(N, A)$ may be written as $\sum_i a_i n_i 1_{mN_0m^{-1}}$ for some finite sets $\{a_i\} \subset A$, $\{n_i\} \subset N$, and some $m \in M^+$, we see that ϕ is completely determined by $\tilde{\phi}$ so (10) is injective.

Take some $\tilde{\phi} \in \mathrm{Hom}_{A[M^+]}(A[M], V^{N_0})$. We can define $\phi \in \mathrm{Hom}_{A[N]}(\mathcal{C}_c^{\mathrm{sm}}(N, A), V)$ as $\phi(f) := \sum_i a_i n_i \tilde{\phi}(m)$ for any $f = \sum_i a_i n_i \tilde{\phi}(m) \in \mathcal{C}_c^{\mathrm{sm}}(N, A)$. By the M^+ invariance of $\tilde{\phi}$, this is well defined independently of the choice of representation for f as such a sum. So ϕ maps to $\tilde{\phi}$ and we have surjectivity. □

(3.29) Proposition. If $U \in \mathrm{Mod}_{M}^{\mathrm{sm}}(A)$, $V \in \mathrm{Mod}_{P}^{\mathrm{sm}}(A)$, then there is a natural isomorphism

$$\mathrm{Hom}_{A[P]}(\mathcal{C}_c^{\mathrm{sm}}(N, U), V) \xrightarrow{\sim} \mathrm{Hom}_{A[M]}(U, \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})).$$

[14] It is clearly injective, and it is surjective as we can extend any $f \in \mathcal{C}_c(N, U)$ to a function on $\overline{P}N$ via $f(\overline{p}n) = \overline{p}f(n)$, setting it to be zero elsewhere on G.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{A[P]}(\mathcal{C}_c^{\mathrm{sm}}(N,U),V) &\xrightarrow{\sim} \mathrm{Hom}_{A[P]}(\mathcal{C}_c^{\mathrm{sm}}(N,A)\otimes_A U,V) \text{ (by (3.27))}\\
&\xrightarrow{\sim} \mathrm{Hom}_{A[P]}(U,\mathrm{Hom}_A(\mathcal{C}_c^{\mathrm{sm}}(N,A),V))\\
&\xrightarrow{\sim} \mathrm{Hom}_{A[M]}(U,\mathrm{Hom}_{A[N]}(\mathcal{C}_c^{\mathrm{sm}}(N,A),V)) \text{ (P action on M factors through P/N)}\\
&\xrightarrow{\sim} \mathrm{Hom}_{A[M]}(U,\mathrm{Hom}_{A[Z_M^+]}(A[Z_M],V^{N_0})) \text{ (by (3.28))}.
\end{aligned}$$

□

We assume the following:

(3.30) Lemma. Let $\phi \in \mathrm{Hom}_{A[P]}(\mathcal{C}_c^{\mathrm{sm}}(N,U),V)$. If $f \in \mathcal{C}_c^{\mathrm{sm}}(N,U)$ and $g \in G$ are such that $gf \in \mathcal{C}_c^{\mathrm{sm}}(N,U)$, then $\phi(gf) = g\phi(f)$.

(3.31) Theorem. If $U \in \mathrm{Mod}_M^{\mathrm{adm}}(A)$ and $V \in \mathrm{Mod}_G^{\mathrm{adm}}(A)$, then taking ordinary parts induces an isomorphism

$$\mathrm{Hom}_{A[G]}(\mathrm{Ind}_P^G U, V) \xrightarrow{\sim} \mathrm{Hom}_{A[M]}(U, \mathrm{Ord}_P(V)).$$

That is, $\mathrm{Ord}_P : \mathrm{Mod}_G^{\mathrm{adm}}(A) \to \mathrm{Mod}_M^{\mathrm{adm}}(A)$ is right adjoint to the functor $U \mapsto \mathrm{Ind}_P^G U$.

(3.32) Remark. The theorem holds replacing $\mathrm{Mod}_M^{\mathrm{adm}}(A)$ by $\mathrm{Mod}_M^{\varpi\text{-adm}}(A)$, and $\mathrm{Mod}_G^{\mathrm{adm}}(M)$ by $\mathrm{Mod}_G^{\varpi\text{-cont}}(A)$.

We shall only prove the theorem in the case that A is Artinian.

Proof of (3.31). Suppose U and V are smooth.

Restricting maps from $\mathrm{Ind}_P^G U$ to $\mathcal{C}_c^{\mathrm{sm}}(N,U)$ induces

$$\begin{aligned}
\mathrm{Hom}_{A[G]}(\mathrm{Ind}_P^G U, V) &\to \mathrm{Hom}_{A[P]}(\mathcal{C}_c^{\mathrm{sm}}(N,U),V)\\
&\xrightarrow{\sim} \mathrm{Hom}_{A[G]}(A[G]\otimes_{A[P]}\mathcal{C}_c^{\mathrm{sm}}(N,U),V)
\end{aligned} \tag{11}$$

which we wish to show is an isomorphism.

The natural map

$$A[G]\otimes_{A[P]}\mathcal{C}_c^{\mathrm{sm}}(N,U) \to \mathrm{Ind}_P^G U$$

is surjective as $\mathcal{C}_c^{\mathrm{sm}}(N,U)$ generates $\mathrm{Ind}_P^G U$ since the G–translates of N cover $\overline{P}\setminus G$, so (11) is injective.

Take $\phi \in \mathrm{Hom}_{A[G]}(A[G]\otimes_{A[P]}\mathcal{C}_c^{\mathrm{sm}}(N,U),V)$. Let $g_1,\ldots,g_l \in G$ and $f_1,\ldots,f_l \in \mathcal{C}_c^{\mathrm{sm}}(N,U)$ be such that $\sum_{i=1}^l g_i f_i = 0$ in $\mathrm{Ind}_P^G U$. We want to show that $\sum_{i=1}^l g_i\phi(f_i) = 0$ in V.

Suppose that each $x \in \overline{P}\setminus G$ has a compact open neighbourhood Ω_x such that for any other neighbourhood $\Omega'_x \subset \Omega_x$ of x, $\sum_{i=1}^l g_i\phi(f_i|_{\Omega'_x g_i}) = 0$. Then we can partition $\overline{P}\setminus G = \amalg_{j=1}^s \Omega'_{x_j}$ and write

$$g_i f_i = \sum_{j=1}^s (g_i f_i)|_{\Omega'_{x_j}} = \sum_{j=1}^s g_i f_i|_{\Omega'_{x_j} g_i}$$

so $f_i = \sum_{j=1}^{s} f_i|_{\Omega'_{x_j}g_i}$ and, if $\sum_{i=1}^{l} g_i\phi(f_i|_{\Omega'_x g_i}) = 0$, then $\sum_{i=1}^{l} g_i\phi(f_i) = 0$ and we are done.

As $x = \overline{P}g$ for some $g \in G$, we can replace $(g_1, \ldots, g_l)$ by $(gg_1, \ldots, gg_l)$, and so we can assume that $x = \overline{P}e$ for e the identity. We can identify this identity coset with the identity of N under the open immersion $N \hookrightarrow \overline{P} \setminus G$. Let Ω_e be an open coset of e in N, so $\Omega_e g_i \subset N$ as open subsets of $\overline{P} \setminus G$ for all i such that $g_i \in \overline{P}N$ and such that $\Omega_e g_i$ is disjoint from the support of f_i for all other i.

Then

$$\sum_{i=1}^{l} (g_i f_i)|_{\Omega_e} = \sum_{i=1}^{l} g_i f_i|_{\Omega_e g_i} = 0$$

in $\mathcal{C}_c^{\mathrm{sm}}(N, U)$.

Note that $f|_{\Omega_e g_i} = 0$ if $g_i \notin N$ by our choice of Ω_e so $g_i f_i \in \mathcal{C}_c^{\mathrm{sm}}(N, U)$ and we can apply ϕ and (3.30) to get $\sum_{i=1}^{l} g_i\phi(f_i|_{\Omega_e g_i}) = 0$, and thus we have shown that it is an isomorphism. We can compose with the isomorphism in (3.29) and we are done in the case that A is Artinian.

Now suppose that U and V are ϖ–adically continuous. There is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{A[G]}(\mathrm{Ind}_{\overline{P}}^{G} U, V) & \longrightarrow & \varprojlim_n \mathrm{Hom}_{(A/\varpi^n)[G]}(\mathrm{Ind}_{\overline{P}}^{G}(U/\varpi^n U), V/\varpi^n V) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{A[M]}(U, \mathrm{Ord}_P(V)) & \longrightarrow & \varprojlim_n \mathrm{Hom}_{A[M]}(U/\varpi^n U, \mathrm{Ord}_P(V/\varpi^n V))
\end{array}$$

where the vertical arrows are given by passing to ordinary parts and the horizontal arrows arise from the isomorphism $(\mathrm{Ind}_{\overline{P}}^{G} U)/\varpi^i(\mathrm{Ind}_{\overline{P}}^{G} U) \cong \mathrm{Ind}_{\overline{P}}^{G}(U/\varpi^i U)$ together with the definition $\mathrm{Ord}_P(V) := \varprojlim_n \mathrm{Ord}_P(V/\varpi^n V)$.

We have proven that the right vertical arrow is an isomorphism so the left vertical arrow also is. $\square$

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