Kato study group: Hasse Weil L-functions

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1 Hasse-Weil L-functions

Recall the L-functions of p-adic Galois representations:

Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ and consider (p, S, V, A) where:

- p is a prime number
- $p \in S$ is a finite set of primes
- V is a finite dimensional vector space over $\mathbb{Q}_p$ endowed with a continuous action of $\mathcal{G}_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which is unramified outside of S.
- A is a commutative ring over $\mathbb{Q}$ which is a finite product of finite extensions of $\mathbb{Q}$ and acts on V in a way that commutes with the action of $\mathcal{G}_{\mathbb{Q}}$.

That is, if $V \xrightarrow{h} V$ is a homomorphism over Λ, then $\det_\Lambda(h : V)$ denotes the element $a \in \Lambda$ such that the map $\det_\Lambda(V) \to \det_\Lambda(V)$ induced by h coincides with multiplication by a. Notice we generally have $\bigwedge^{\dim V} V \xrightarrow{\cdot a = \det h} \bigwedge^{\dim V} V$.

Recall $\det_\Lambda(V) = L$ where

$$
\begin{aligned}
r &: \mathrm{Spec}\,\Lambda \to \mathbb{Z} \\
\mathfrak{p} &\mapsto \mathrm{rk}_{\Lambda_\mathfrak{p}}(V \otimes_\Lambda \Lambda_\mathfrak{p}) \\
L &= \bigoplus_{\mathfrak{p}} \bigwedge_{\mathfrak{p}}^{r(\mathfrak{p})} V \otimes \Lambda_\mathfrak{p}
\end{aligned}
$$

Definition 1.1. Let $\Lambda = A \otimes \mathbb{Q}_l$ for prime $l \notin S$ and σ_l be the arithmetic Frobenius of l in $\mathcal{G}_{\mathbb{Q}}$ and define:

$$P_{\Lambda,l}(V, t) := \det_\Lambda(1 - \sigma_l^{-1} t : V) \in \Lambda[t]$$

where $\det_\Lambda$ is the determinant over Λ.

Definition 1.2. Assume the following conditions on (p, S, V, A) are satisfied:

1. For any prime number $l \notin S$, $P_{\Lambda,l} \in A[t]$.

2. The product
$$L_{A,S}(V,s)=\prod_{l\notin S}P_{\Lambda,l}(V,l^{-s})^{-1}$$
with $s\in\mathbb{C}$ converges absolutely in $A\otimes\mathbb{C}$ for $\mathrm{Re}(s)\gg 0$

We call the $A\otimes\mathbb{C}$-valued function $L_{A,s}(V,s)$ (where s is such that it converges) the *L-function of V with respect to A and S.*

If $A=\mathbb{Q}$ then we write $L_{A,S}(V,s)=L_S(V,s)$

Remark 1.3.

- *If (p,S,V,A) satisfies $(1),(2)$ and $r\in\mathbb{Z}$, then $(p,S,V(r),S)$ also satisfies $(1),(2)$ (where $V(r)$ is the r-fold Tate twist of V), and*
$$L_{A,S}(V(r),s)=L_{A,S}(V,s+r).$$
- *If there is an exact sequence*
$$0\to V'\to V\to V''\to 0$$
with common (p,S,A) satisying $(1),(2)$, we have
$$L_{A,s}(V,s)=L_{A,S}(V',s)L_{A,s}(V'',s).$$

We want to refine this definition incluse the missing 'bad reduction' factors. 'bad' primes carry good information (theory of conductor). We want to correct at the ramified primes by taking the largest quotient of the representation on which the inertia group acts trivially.

Definition 1.4. We now wish to define $P_{\Lambda,l}(V,t)\in\Lambda[t]$ for $l\in S$.

Let $\bar{\mathbb{Q}}_l$ be an algebraic closure of $\mathbb{Q}_l$, take an embedding $\bar{\mathbb{Q}}\to\bar{\mathbb{Q}}_l$ and consider the representations of $\mathcal{G}_{\mathbb{Q}_l}$ in V via the induced map $\mathcal{G}_{\mathbb{Q}_l}\to\mathcal{G}_{\mathbb{Q}}$. We define:
$$P_{\Lambda,l}(V,t)=\begin{cases}\det_\Lambda(1-\sigma_l^{-1}t\,:\,H^0(\mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_{l,ur}),V))=\det_\Lambda(1-\sigma_l^{-1}t\,:\,V^{I_{\bar{\mathbb{Q}}_l/\mathbb{Q}_l}}) & \text{if } l\neq p\\ \det_\Lambda(1-\varphi_l^{-1}t\,:\,D_{crys}(V)) & \text{if } l=p\end{cases}$$

where $\mathbb{Q}_{l,ur}$ is the maximal unramified extension of $\mathbb{Q}_l\subset\bar{\mathbb{Q}}_l$, $\sigma_l\in\mathrm{Gal}(\mathbb{Q}_{l,ur}/\mathbb{Q}_l)$ denotes the arithmetic Frobenius, D_{crys} is a functor of Fontaine in chapter 2, 1.2.1, and φ_l is the Frobenius operator.

$P_{\Lambda,l}(V,t)$ is independent of the choices of $\bar{\mathbb{Q}}_l$ and $\bar{\mathbb{Q}}\to\bar{\mathbb{Q}}_l$.

If $P_{\Lambda,l}(V,t)\in A[t]$ for all $l\in S$ and $(1),(2)$ are satisfied, then we define
$$L_A(V,s)=\prod_l P_{\Lambda,l}(V,l^{-s})^{-1}$$
where l ranges over all prime numbers (write $L(V,s)$ if $A=\mathbb{Q}$).

From here on just consider $A=\mathbb{Q}$.

1.1 Hasse-Weil L-function

Definition 1.5. Let X be a proper smooth scheme over $\mathbb{Q}$ and p a prime number. Let
$$V_m = H^m_{et}(X \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p)$$
and let $S = \{p\} \cup \{\text{primes at which } X \text{ has bad reduction}\}$.

(Here the l-adic cohomology group $H^i(V, \mathbb{Z}_l) := \varprojlim H^i(V, \mathbb{Z}/l^k\mathbb{Z})$ and then we remove tosion and get cohomology groups that are vector spaces of characteristic 0 by defining $H^i(V, \mathbb{Q}_l) := H^i(V, \mathbb{Z}_l) \otimes \mathbb{Q}_l$)

Then $(p, S, V_m, \mathbb{Q})$ satisfies conditions $(1), (2)$ and the functions
$$L_S(H^m(X), s) := L_S(V_m, s)$$
for $m \in \mathbb{Z}$ are called the *Hasse-Weil L-functions of X* and are independent of p.

Conjecture 1.6.

- *These functions have analytic continuations to $\mathbb{C}$ as meromorphic functions.*
- *$L(V_m, s)$ is denoted by $L(H^m(X), s)$ if $P_{\mathbb{Q},l}(V_m, t) \in \mathbb{Q}[t]$ and is independent of p for any $l \in S$. These conditions are satisfied, for example, in the case that X is an abelian variety over $\mathbb{Q}$ and are conjectured to always hold.*

Remark 1.7. *These definitions of L-functions are generalized in a natural way to p-adic representations V of $\mathcal{G}_K$ and to proper smooth schemes X/K for number fields K (instead of σ_l, we consider the arithmetic Frobenius of a non-zero prime ideal in $\mathcal{O}_K$).*

However, the L-function of such V coincides with the L-function of the p-adic representation of $\mathcal{G}_{\mathbb{Q}}$ induced from V (respectively the L-function of X viewed as a scheme over $\mathbb{Q}$ via $X \to \operatorname{Spec} K \to \operatorname{Spec} \mathbb{Q}$).

We consider mainly the case $K = \mathbb{Q}$ in the paper.

Example 1.8. *Let E be an elliptic curve over $\mathbb{Q}$.*

Notice in the case $m = 1$, the p-adic Tate module $\varprojlim E[p^n]$ is the dual of $H^1_{et}(E \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_p)$ and higher m correspond to exterior powers.

Then $L(E, s) = \prod_p L_p(E, s)^{-1}$ where for a given prime p:
$$L_p(E, s) = \begin{cases} 1 - a_p p^{-s} + p^{1-2s} & \textit{if } E \textit{ has good reduction at } p \\ 1 - a_p p^{-s} & \textit{if } E \textit{ has multiplicative reduction at } p \\ 1 & \textit{if } E \textit{ has additive reduction at } p \end{cases}$$
where $a_p = p + 1 - E(\mathbb{F}_p)$ in the good reduction case and in the multiplicative reduction case $a_p = \pm 1$ according to whether it is split or nonsplit.

More, generally for abelian varieties we can describe the l-adic cohomology as the dual of the Tate module and its exterior powers. For curves the first cohomology group is the first cohomology group of its Jacobian. That is, there is an isomorphism of $\mathcal{G}_K$ modules $H^1_{et}(X_{\bar{K}}, \mathbb{Q}_p) \cong T_p(J(X)) \otimes \mathbb{Q}_p$ for X a smooth projective algebraic curve over a number field K.

2 Relation to de Rham cohomology

Definition 2.1. Suppose the A is an abelian category with enough injectives, B is another abelian category, and F is a left exact functor. If $C^\bullet$ is a complex of objects of A bounded on the left, the *hypercohomology* $\mathbf{H}^i(C^\bullet)$ of $C^\bullet$ for $i \in \mathbb{Z}$ is calculated as follows:

1. Take an quasi-isomorphism $\phi : C^\bullet \to I^\bullet$ for $I^\bullet$ a complex of injectives in A.
2. Then the hyper cohomology $\mathbf{H}^i(C^\bullet) = H^i(F(I^\bullet))$ the normal cohomology.

Equivalently it is the $RF(C^\bullet)$ considered as an element of the derived category of B.

Can read about the following on the Stacks project:

Definition 2.2. For a smooth proper variety X over a field K, the *de Rham cohomology groups* $H^m_{dR}(X)$ are defined by using differential forms on X to be the hyper-cohomology groups $H^m(X, \Omega_X)$ where Ω_X is the de Rham complex

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots$$

For a smooth scheme $X/\mathbb{C}$ there exists a canonical isomorphism

$$H^m_{dR}(X/\mathbb{C}) \cong H^m(X(\mathbb{C}), \mathbb{C}) \tag{1}$$

between the de Rham (differential forms) and singular (Betti) cohomologies (here $X(\mathbb{C})$ has the classical topology). This is called the *period isomorphism* (integration of algebraic differential forms over topological cycles).

Remark 2.3. *On the other hand, for a smooth scheme X over a complete discrete valuation field K with perfect residue field k such that* $\operatorname{char} K = 0$ *and* $\operatorname{char} k = p > 0$*, there exists a canonical isomorphism*

$$H^m_{dR}(X/K) \otimes_K B_{dR} \cong H^m_{et}(X \otimes_K \bar{K}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR}$$

(here B_{dR} is a large field defined by Fontaine) which gives a relationship between differential forms and p-adic etale cohomology. This is analagously called the p-adic period isomorphism.

We wish to explain how the relationship (1) is related to values of Hasse-Weil L-functions.

Example 2.4. *Consider the elliptic curve $E/\mathbb{Q}$:*

$$E = \{(x, y) : y^2 = x^3 + 1\} \cup \{\infty\}$$

and let $X = E \otimes \mathbb{C}$. The isomorphism for $m = 1$ is

$$\begin{array}{ccccc} H^1_{dR}(X/\mathbb{C}) & \xrightarrow{\sim} & H^1(X(\mathbb{C}), \mathbb{C}) & = & \hom(H_1(X(\mathbb{C}), \mathbb{Q}), \mathbb{C}) \\ \cup & & & & \\ H^0(X, \Omega^1_{X/\mathbb{C}}) \ni \omega & \mapsto & (\gamma \mapsto \int_\gamma \omega) & & \end{array}$$

which indicates why the isomorphism is called the period isomorphism.

If we put

$$\omega = \frac{dx}{y} \in H^0(E, \Omega^1_{E/\mathbb{Q}}) \subset H^0(E, \Omega^1_{E/\mathbb{Q}}) \otimes \mathbb{C} = H^0(X, \Omega^1_{X/\mathbb{C}})$$

$$\gamma = (\textit{the loop } E(\mathbb{R}) \textit{ orientated from upper to lower in the plane}) \in H_1(X(\mathbb{C}), \mathbb{Q})$$

which are a rational differential and rational homology class respectively, then (not obviously)

$$L(H^1(E), 1) = 12^{-1} \int_\gamma \omega.$$

Such a rationality property is generalised to a conjecture of Deligne and to a conjecture of Beilinson.

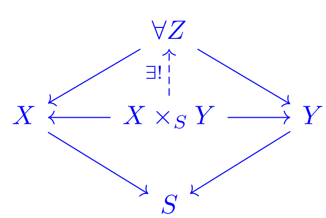
A well known principle in number theory is that, in studying objects over a number field, local theories over local fields play important roles.

We expect the following:

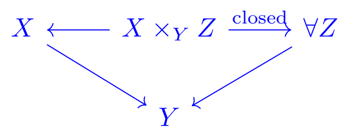
Zeta values are related to differential forms via the theory of period integrals, and then related to etale cohomology theory via the theory of p-adic periods.

A Appendix

Definition A.1. We define the fibered product as follows:



A morphism $f : X \to Y$ of schemes is *universally closed* if for every scheme Z with a morphism $Z \to Y$, the projection $X \times_Y Z \to Z$ is a closed map of the underlying topological spaces.



A morphisms of schemes f is *separated* if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immmersion.

A morphism of schemes $f : X \to Y$ is of *finite type* if there exists a covering of Y by open sets $V_i = \operatorname{Spec} B_i$ and finitely many open affine sets $U_{ij} = \operatorname{Spec} A_{ij}$ such that for each i U_{ij} cover $f^{-1}(V_i)$ and each A_{ij} is finitely generated as a B_i algebra.

A morphism of schemes is *proper* if it is separated, universally closed, and of finite type.

All projective morphisms are proper.

Definition A.2. A *proper variety* is an algebraic variety such that the morphism $X \to \operatorname{Spec} k$ is proper.