Representation Theory of Finite Groups

Abigail Burton

30th September 2019

Motivation

Representation Theory is the study of algebraic structures such as groups, algebras and Lie algebras by representing their elements as linear transformations of vector spaces.
It is useful as it reduces problems in algebra to problems in linear algebra which are better understood, for example, we can often extend representations to Hilbert spaces and then apply tools in analysis.
In this talk I aim to present key definitions and a few simple examples, outline some important structure theorems and then end with a a short introduction to character theory.

Key Definitions

- A **representation of a finite group** G is a pair (ρ, V) for V a vector space and $\rho : G \to \mathrm{GL}(V)$ a group homomorphism.
- That is, for all $g \in G$, $\rho(g) : V \to V$ is a linear map and for all $g, h \in G$, $\rho(gh) = \rho(g) \circ \rho(h)$ and $\rho(1_G) = \mathrm{id}_V$.
- A **representation of an algebra** A is a pair (ρ, V) for V a vector space and $\rho : A \to \mathrm{End}_k V$ an algebra homomorphism. This is the same notion as 'module' with $a \cdot v := \rho(a)v$.
- It is easy to see that each representation of the group G corresponds to a kG-module and vice versa (in fact this is an isomorphism of categories).
- We say that two representations $\rho_i : A \to \mathrm{End}_k(V_i)$ for $i = 1, 2$ are **equivalent** if there exists a linear isomorphism $\psi : V_1 \to V_2$ such that $\rho_1(a) = \psi^{-1} \circ \rho_2(a) \circ \psi$ for all $a \in A$.
- In the category of representations, the morphisms $(\rho_1, V_1) \to (\rho_2, V_2)$ are maps $T : V_1 \to V_2$ such that $\rho_2(g) \circ T = T \circ \rho_1(g)$ holds for all $g \in G$. These are called G-linear maps.

Examples

- We always have the trivial representation $\rho(g) = \mathrm{id}_V \ \forall\, g \in G$ where $V = k$.
- If $G = D_{2n} = \langle r, \sigma \mid r^n = \sigma^2 = 1,\, \sigma r \sigma^{-1} = r^{-1} \rangle$ is the dihedral group, then for each $1 \leq m \leq n-1$ there is a natural representation $\rho_m : G \to \mathrm{GL}(\mathbb{R}^2)$ given by:

$$\rho_m(r) = \begin{pmatrix} \cos(\frac{2\pi m}{n}) & -\sin(\frac{2\pi m}{n}) \\ \sin(\frac{2\pi m}{n}) & \cos(\frac{2\pi m}{n}) \end{pmatrix} \ \rho_m(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- If G acts on a set Ω, we may define the permuation module $k\Omega$ as the k-vector space with basis $\{\omega \in \Omega\}$ with G acting as $g \cdot \left(\sum_{\omega\in\Omega} \lambda_\omega \omega\right) = \sum_{\omega\in\Omega} \lambda_\omega (g \cdot \omega)$.
- $M = \{(x_1, \ldots, x_n) \in k^n \mid x_1 + \cdots + x_n = 0\}$ is a kS_n module under permutation on indices. In fact this is a simple module.
- $M_n(k)$ has a unique simple module k^n.

More Definitions

- A representation $\rho : G \to GL(V)$ is **reducible** if there is a proper subspace $0 \neq W \subset V$ such that $\rho(g)W \subseteq W$ for all $g \in G$ (a **subrepresentation**). Otherwise we say it is **irreducible**.
- In the algebra case, we define **simple** modules as those with no nontrivial proper submodules.
- A module M is **semisimple** if $M = \bigoplus_{i \in I} S_i$ for some simple submodules S_i.
- A **composition series** for a module M is a sequence of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ such that each **composition factor** M_{i+1}/M_i is a simple module.
- **Example:** If G is a finite group acting transitively (meaning that it has a single orbit) on a finite set Ω, then the permutation representation $k\Omega$ for $\text{char}(k) = p$ is a semisimple kG-module iff $p \nmid |\Omega|$.

Key Theorems

- **Schur's Lemma:** If $f : M \to N$ is a homomorphism of simple A-modules then either $f = 0$ or f is an isomorphism. Further, if k is algebraically closed and $M = N$ is finite dimensional, then $f = \lambda \mathrm{Id}_M$.
- **Artin-Wedderburn Theorem:** Let A be a finite dimensional k-algebra with k algebraically closed. Then A is semisimple iff $A \cong M_{n_1}(k) \times \cdots \times M_{n_s}(k)$ for some unique integers $n_i \in \mathbb{N}$.
- **Jordan-Hölder Theorem:** Let M be a non-zero finite dimensional A-module. Then M has a composition series and all composition series are equivalent (have the same length and same composition factors with multiplicity).
- **Maschke's Theorem.** Let G be a finite group and k a field. Then kG is semisimple iff $\mathrm{char}(k) \not| |G|$. In particular, $\mathbb{C}G$ is semisimple.

Useful Corollaries

- **Corollary to Schur:** Every simple finite dimensional module of a commutative algebra of an algebraically closed field is one dimensional.
- **Corollary to Jordan Hölder:** A finite dimensional algebra A has only finitely many isomorphism classes of simple A-modules.
- **Corollary to Artin-Wedderburn:** For A a finite dimensional semisimple k-algebra, $A \cong \prod_{i=1}^{s} M_{n_i}(k)$. Then
  1. A has exactly s simple modules $M_1, \ldots, M_s$ with $\dim_k M_i = n_i$.
  2. $s = \dim_k Z(A)$ (as $Z(A) \cong \prod Z(M_{n_i}(k)) \cong \prod k \cdot \mathrm{Id}_{n_i} = k^s$)
  3. $\dim_k A = n_1^2 + \cdots + n_s^2$.
- **Corollary to above and Maschke:** For k algebraically closed with $\mathrm{char}(k) \not| |G|$, kG has exactly s nonisomorphic simple modules, where s is the number of conjugacy classes. If $n_1, \ldots, n_s$ are the dimensions of the simple modules, then $|G| = n_1^2 + \cdots + n_s^2$.

Characters

- If (ρ, V) is a finite dimensional representation of G. The **character** of the representation is the function $\chi_V : G \to k$ is defined by $\chi_V(g) = \mathrm{tr}\rho(g)$.
- We have the following results:
  1. If V_1, V_2 are equivalent repressentations then $\chi_{V_1} = \chi_{V_2}$
  2. $\chi_V(e) = \dim V$
  3. χ is constant on conjugacy classes
  4. If $k = \mathbb{C}$ and $g \in G$ has finite order then $\chi_V(g^{-1}) = \overline{\chi_V(g)}$
  5. $\chi_{V \oplus W} = \chi_V + \chi_W$
  6. $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
  7. $\chi_{V^*}(g) = \chi_V(g^{-1})$
  8. $\chi_V(g)$ is an algebraic integer
  9. $\chi_U = \chi_V \iff U \cong V$
- **Fixed Point Formula:** If $|G|$ is invertible in k, then $\dim U^G = \frac{1}{|G|} \sum_{g \in G} \chi_U(g)$.

More on Characters

- We can define an **inner product** $\langle\, ,\, \rangle$ on $\mathcal{C}_{\text{class}}(G)$ (functions $G \to k$ constant on conjugacy classes) by $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g^{-1}) f_2(g)$.
- This has the following properties:
  1. For V, W G-representations, $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W)$
  2. $\langle \chi_V, \chi_W \rangle = 0$ if $V \not\cong W$
  3. $\langle \chi_V, \chi_W \rangle = 1$ if $V = W$ and k is algebraically closed
  4. If k is algebraically closed and $\mathrm{char}(k) \not| \, |G|$ then $\{\chi_V\}_{V \text{ irreducible}}$ is an orthonormal basis of $\mathcal{C}_{\text{class}}(G)$.
- We define the **character table** of G as the finite square matrix $A = (\chi_i(C_j))$ for C_j the conjugacy classes of G. This has the following useful relations:
  1. **Orthogonality of characters:** $\overline{A} D A^t = I$ for D diagonal with entries $d_j = \frac{|C_j|}{|G|}$
  2. **Orthogonality of columns:** $A^t A = D^{-1}$
- If we have a character χ_U and V_i are the irreducible representations then if $\chi_U = \sum_i n_i \chi_{V_i}$:
  1. $n_i = \langle \chi_U, \chi_{V_i} \rangle$
  2. $\langle \chi_U, \chi_U \rangle = \sum n_i^2$ so U is irreducible iff $\langle \chi_U, \chi_U \rangle = 1$

Thanks for coming!