Question: Can we p–adically interpolate the space of modular forms? (Give a cusp form, can we find a family of modular forms containing it that varies p–adic analytically over weights?

Hida Theory: Yes, in the case of ordinary parts.

1 Hida Theory

Haruzo Hida began to develop his theory of ordinary parts in the 1980's. We will give a brief summary of his work before moving onto an adaptation of one of his proofs by Emerton in 1999.

Let $\Lambda = \mathbb{Z}_p[[X]](= \mathbb{Z}_p[[\Gamma]]$ under $\gamma \mapsto (1+X))$. Let γ be a topological generator for $1+p\mathbb{Z}_p$ (say, $1+p$), and $\omega : \mathbb{Z}_p^\times \to (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$. For $\zeta \in \mu_{p^{r-1}}$, let $\chi_\zeta : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \mathbb{C}^\times$ be defined by $\chi_\zeta(\gamma) = \zeta$.

(1.1) Definition (Λ-adic forms)**.** A Λ-adic form F of level N and character $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathbb{C}^\times$ is a formal q-expansion $F = \sum_{n=0}^\infty a_n(X)q^n \in \Lambda[[q]]$ such that for all $k \geq 2$, $\zeta \in \mu_{p^{r-1}}$ with $r \geq 1$, the q-expansion $f = \sum_{n=0}^\infty a_n(\zeta\gamma^k - 1)q^n \in M_k(\Gamma_1(Np^r), \chi\omega^{-k}\chi_\zeta; \mathbb{Z}_p) \subset \overline{\mathbb{Q}}_p[[q]]$.

That is, it is the image under a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ of the q-expansion in $\overline{\mathbb{Q}}[[q]]$ of a classical modular form of weight k, level Np^r, character $\chi_v = \chi\omega^{-k}\chi_\zeta$.

We write $M(N, \chi, \Lambda)$ for the Λ-module of Λ-adic forms of tame level N and character χ. Let $M(N, \Lambda) = \bigoplus_\chi M(N, \chi, \Lambda)$ be the Λ-module of Λ-adic forms. We can similarly define $S(N, \Lambda)$ as the space of cusp forms.

The idea is that we get a family of modular forms of varying weights and levels, all with the same residual q-expansion.

These spaces have a natural action of the Hecke operators U_q.

If f is a U_p-eigenform, then we say it is p-ordinary if $U_p(f) = a_p(f)f$ for $a_p(f) \in \mathbb{Z}_p^\times$. However, we need that the sum of ordinary forms is ordinary, so Hida defined $e = \lim_{n\to\infty} U_p^{n!}$, the ***ordinary projector***.

Then define $M^{\text{ord}}(N, \Lambda) = eM(N, \Lambda)$, and similarly for cusp forms and Hecke algebras.

There are the following two important theorems. The first gives a result on the independence of weight, and the second shows that this is the interpolating space we want. We will prove the second.

(1.2) Theorem. $S(N, \Lambda)^{\text{ord}}$ is a free Λ-module of finite rank and for all $k \geq 2$

$$\begin{aligned} \operatorname{rank}_\Lambda S^{\text{ord}}(N, \Lambda) &= \operatorname{rank}_{\mathbb{Z}_p} S_k^{\text{ord}}(\Gamma_1(Np^r), \chi\omega^{-k}; \mathbb{Z}_p) \\ &= \operatorname{rank}_{\mathbb{Z}_p} S_2^{\text{ord}}(\Gamma_1(Np^r), \chi\omega^{-2}; \mathbb{Z}_p). \end{aligned}$$

$S_k^{\text{ord}}(\Gamma_1(\chi); \mathbb{Q}_p) = S_k^{\text{ord}}(\Gamma_1(N); \mathbb{Z}_p) \otimes \mathbb{Q}_p$, so the number of "distinct" p-ordinary cuspidal eigenforms in $S_k(\Gamma_1(N); \mathbb{Q}_p)$ is constant for $k \geq 2$ as they form a basis of S_k.

(1.3) Theorem. If $\mathfrak{p}_s = (\gamma^k - 1) \subset \Lambda$, then for all $k \geq 2$

$$S^{\text{ord}}(\chi, \Lambda)/\mathfrak{p}_k \cong S_k^{\text{ord}}(\Gamma_1(p), \chi\omega^{-k}; \mathbb{Z}_p).$$

2 Emerton's Proof

In 1999, Emerton gave representation theoretic proofs of the above theorems.

Let $p \geq 5$ and let $(N, p) = 1$ be such that $\Gamma_1(Np)$ is torsion free.

Emerton considers the tower of modular curves

$$\cdots \to Y_1(Np^r) \to \cdots \to Y_1(Np) \tag{1}$$

corresponding to the chain of congruence subgroups

$$\cdots \subset \Gamma_1(Np^r) \subset \cdots \subset \Gamma_1(Np)$$

Taking the homology with $\mathbb{Z}$–coefficients of the chain of modular curves, we get a chain

$$\cdots \to \Gamma_1(Np^r)^{\mathrm{ab}} \to \cdots \to \Gamma_1(Np)^{\mathrm{ab}}. \tag{2}$$

(2.1) Definition (Principal units). Let $\Gamma_r = 1 + p^r\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ be the subgroup of index p^{r-1} in $\Gamma := \Gamma_0$

(2.2) Definition (Intermediate congruence subgroups). For $r \geq s > 0$, define congruence subgroups

$$\Phi_r^s = \Gamma_1(Np^s) \cap \Gamma_0(p^r) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \mod Np^s, c \equiv 0 \mod Np^r \right\}$$

with $\Gamma_1(Np^r) \subset \Phi_r^s \subset \Phi_r^1 \subset \Gamma_1(Np)$.

(2.3) Remark. There is surjection

$$\Phi_r^s \to \Gamma_s/\Gamma_r$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod p^r.$$

Φ_r^s satisfies

$$1 \to \Gamma_1(Np^r) \to \Phi_r^s \to \Gamma_s/\Gamma_r \to 1,$$

so $\Phi_r^s/\Gamma_1(Np^r) \cong \Gamma_s/\Gamma_r$ and abelianisation gives us

$$\Gamma_1(Np^r)^{\mathrm{ab}} \to \Phi_r^{s\,\mathrm{ab}} \to \Gamma_s/\Gamma_r \to 1$$

which is not a short exact sequence, however, the following is

$$1 \to \Gamma_1(Np^r)^{\mathrm{ab}}/\mathfrak{a}_s \to \Phi_r^{s\,\mathrm{ab}} \to \Gamma_s/\Gamma_r \to 1. \tag{3}$$

where $\mathfrak{a}_s$ denotes the augmentation ideal of $\mathbb{Z}[\Gamma_s]$.

This implies in particular that a map $\Gamma_1(Np^r)^{\mathrm{ab}} \to \Gamma(Np^s)^{\mathrm{ab}}$ in (2) can be factored as the composition

$$\Gamma_1(Np^r)^{\mathrm{ab}} \to \Gamma_1(Np^r)^{\mathrm{ab}}/\mathfrak{a}_s \to \Phi_r^{s\,\mathrm{ab}} \to \Gamma_1(Np^s)^{\mathrm{ab}}. \tag{4}$$

(2.4) Definition (The nebentypus action). The action of Φ_r^1 on $\Gamma_1(Np^r)$ by conjugation induces an action of the quotient $\Phi_r^1/\Gamma_1(Np^r) \cong \Gamma/\Gamma_r$ on $\Gamma_1(Np^r)^{\mathrm{ab}}$. Thus Γ acts on $\Gamma_1(Np^r)^{\mathrm{ab}}$ through Γ/Γ_r and the morphisms in the chain (2) are Γ–equivariant. We call the morphisms induced by the elements of Γ the ***diamond operators***.

(2.5) Definition (Hecke operators)**.** Suppose that T is a group with $G, H \leq T$ and suppose that $t \in T$ is such that $t^{-1}Ht \cap G$ has finite index in G.

$$G^{\mathrm{ab}} \xrightarrow{V} (t^{-1}Ht \cap G)^{\mathrm{ab}} \xrightarrow{t(-)t^{-1}} (H \cap tGt^{-1})^{\mathrm{ab}} \longrightarrow H^{\mathrm{ab}}, \qquad [t] : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$$

Here V is a transfer morphism and the final map is induced by inclusion.

(2.6) Definition (Atkin U–operator)**.** Take $T = \mathrm{GL}_2(\mathbb{Q})$ and $G = H = \Gamma(Np)$ some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then if $t = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ we denote the corresponding Hecke operator by $U = [t]$, the ***Atkin U–operator***.

(2.7) Remark. Suppose $G = \Phi_r^s$ in (2.6). Then

$$t^{-1}\Phi_r^s t \cap \Phi_r^s = \Phi_r^s \cap \Gamma^0(p)$$
$$\Phi_r^s \cap t\Phi_r^s t^{-1} = \Phi_{r+1}^s$$

so U is defined by the composition

$$U : \Phi_r^{s\,\mathrm{ab}} \xrightarrow[V]{} (\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \xrightarrow[t(-)t^{-1}]{} \Phi_{r+1}^{s\,\mathrm{ab}} \longrightarrow \Phi_r^{s\,\mathrm{ab}}, \qquad U' : \Phi_r^{s\,\mathrm{ab}} \to \Phi_{r+1}^{s\,\mathrm{ab}}.$$

With this action, $\Phi_r^{s\,\mathrm{ab}}$ can be made into a $\mathbb{Z}[U]$–module and the maps $\Phi_r^{s\,\mathrm{ab}} \to \Phi_{r'}^{s'\,\mathrm{ab}}$ for $r \geq s > 0, r' \geq s' > 0, r \geq r', s \geq s'$ induced by inclusions are morphisms of $\mathbb{Z}[U]$–modules.

Further, the action of U on $\Phi_r^{s\,\mathrm{ab}}$ commutes with the action of Γ on $\Phi_r^{s\,\mathrm{ab}}$ via the diamond operators defined in (2.4).

(2.8) Remark. $\Gamma_1(Np^r)^{\mathrm{ab}} \to \Phi_r^{s\,\mathrm{ab}}$ is a morphism of $\mathbb{Z}[U]$–modules and so its cokernel Γ_s/Γ_r is also a $\mathbb{Z}[U]$–module. Further, U acts on Γ_s/Γ_r as multiplication by p.

(2.9) Definition (Ordinary parts of modules)**.** Let W be a $\mathbb{Z}_p[U]$–module, finitely generated as a $\mathbb{Z}_p$–module.

If A is the image of the morphism of $\mathbb{Z}_p$–modules $\mathbb{Z}_p[U] \to \mathrm{End}_{\mathbb{Z}_p}(W)$, then as $\mathrm{End}_{\mathbb{Z}_p}(W)$ is a finite $\mathbb{Z}_p$–algebra, A is also a finite $\mathbb{Z}_p$–algebra. Any finite $\mathbb{Z}_p$-algebra factors as a project of local rings, so A is a product of its localizations at finitely many ideals $A = \prod_{\mathfrak{m}\ \max} A_{\mathfrak{m}}$.

The projection $U_{\mathfrak{m}}$ of U onto $A_{\mathfrak{m}}$ will either be contained in the maximal ideal $\mathfrak{m}$ of $A_{\mathfrak{m}}$ or is a unit, we say that $\mathfrak{m}$ is ***ordinary*** if the latter holds.

Let $A^{\mathrm{ord}} := \prod_{\mathfrak{m}\ \mathrm{ordinary}} A_{\mathfrak{m}}$. Each of these is a flat A–algebra as it is projective and a subalgebra of $\mathrm{End}_{\mathbb{Z}_p}(W)$ so taking ordinary parts is exact.

Finally, we define $W^{\mathrm{ord}} := W \otimes_A A^{\mathrm{ord}}$, the ***ordinary part of*** W.

Now, taking U to be Atkin's U–operator corresponding to p, we may consider the ordinary part of the homology group $H_1(Y_1(Np^r), \mathbb{Z}_p) = \Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p$. Notice that $(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$ is a Γ–module since the action of Γ commutes with U.

(2.10) Proposition. If $r \geq s > 0$, then there is an isomorphism of abelian groups

$$(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s \xrightarrow{\sim} (\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}.$$

Proof. Take $U' : \Phi_{r-1}^{s,\mathrm{ab}} \to \Phi_r^{s\,\mathrm{ab}}$ as before. If $\pi : \Phi_r^{s\,\mathrm{ab}} \to \Phi_{r-1}^{s\,\mathrm{ab}}$ is the map induced by inclusion, then $U' \circ \pi = U \in \mathrm{End}(\Phi_r^{s\,\mathrm{ab}})$ and $\pi \circ U' = U \in \mathrm{End}(\Phi_{r-1}^{s\,\mathrm{ab}})$.

If $\mathfrak{m}$ is ordinary, then $U_{\mathfrak{m}}$ is a unit in $\mathrm{End}_{\mathbb{Z}_p}(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)_{\mathfrak{m}}$ so we can take $U^{-1} = \prod_{\mathfrak{m}\text{ ordinary}} U_{\mathfrak{m}}^{-1}$ acting on $(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z})^{\mathrm{ord}}$. Then π induces an isomorphism

$$(\Phi_r^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \cong (\Phi_{r-1}^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$$

with inverse $U^{-1} \circ U'$.

Inductively, we obtain

$$(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \cong (\Phi_s^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \cong (\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}.$$

On tensoring by $\mathbb{Z}_p$ and taking ordinary parts in the sequence (3), we get:

$$1 \to (\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s \to (\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \to (\Gamma_s/\Gamma_r)^{\mathrm{ord}} \to 1.$$

This is as Γ_s/Γ_r is p–torsion so $\Gamma_s/\Gamma_r \otimes \mathbb{Z}_p = \Gamma_s/\Gamma_r$. Also, U is Γ–equivariant so taking Γ_s–coinvariants (i.e. quotienting by $\mathfrak{a}_s$) and taking ordinary parts are commuting functors. By (2.8) U is nilpotent on Γ_s/Γ_r and so $(\Gamma_s/\Gamma_r)^{\mathrm{ord}} = 1$ which implies that

$$(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s \cong (\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}.$$

□

(2.11) Definition (Iwasawa module). We may take a projective limit in the chain of $\mathbb{Z}_p$–modules

$$\mathbf{W} := \varprojlim_r (\cdots \to \Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p \to \cdots \to \Gamma_1(Np)^{\mathrm{ab}} \otimes \mathbb{Z}_p).$$

The profinite group Γ acts on $\Gamma_1(Np^r) \otimes \mathbb{Z}_p$ through its finite quotient Γ/Γ_r and so $\mathbf{W}$ is a module over the completed group algebra $\Lambda := \varprojlim_r \mathbb{Z}_p[\Gamma/\Gamma_r]$.

(2.12) Proposition. Suppose that M_r is a projective system of Λ–modules such that the M_r are invariant under Γ_r and such that for any $r \geq s$, the morphisms factor as $M_r \to M_r/\mathfrak{a}_s \to M_s$. Suppose further that each M_r is p–adically complete and the morphisms $M_r/\mathfrak{a}_s \to M_s$ are isomorphisms.

Let $\mathbf{M} = \varprojlim_r M_r$.

Then for any s, the morphism $\mathbf{M}/\mathfrak{a}_s \to M_s$ is an isomorphism.

In particular, this together with (2.10) and (4), implies that

(2.13) Theorem. For any $r \geq 1$, we have that the Γ_r–coinvariants of $\mathbf{W}^{\mathrm{ord}}$ are equal to $H_1(Y_r, \mathbb{Z}_p)^{\mathrm{ord}}$

$$\begin{aligned}\mathbf{W}^{\mathrm{ord}}/\mathfrak{a}_r\mathbf{W}^{\mathrm{ord}} &= (\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \\ &= H_1(Y_1(Np^r), \mathbb{Z}_p)^{\mathrm{ord}}.\end{aligned}$$

This theorem can be used to show that $\mathbf{W}^{\mathrm{ord}}$ is a finitely generated Λ-module. Emerton goes on to prove, using the basical algebraic topology of $Y_1(Np^r)$, and the fact that any finitely generated reflexive Λ-module is free:

(2.14) Theorem. $\mathbf{W}^{\mathrm{ord}}$ is a free Λ-module of finite rank.